Variational proof of the existence of the super-eight orbit in the four-body problem

Mitsuru Shibayama
Osaka University

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N-body problem

The motion of N particles in $\mathbb{R}^d$ under the effects of gravity is governed by the differential equations

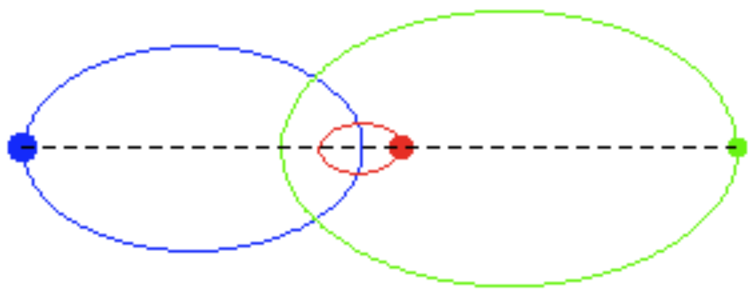
$$m_k \frac{d^2 q_k}{dt^2} = -\sum_{j \neq k} \frac{m_k m_j}{|q_k - q_j|^3}(q_k - q_j) \qquad (q_k \in \mathbb{R}^d,\ \ k = 1, 2, \ldots, n)$$

where $m_k > 0$ is the k-th mass.

A fundamental problem is to find periodic solutions!

Classical solutions

Euler solution(1767)



Lagrange solution(1772)

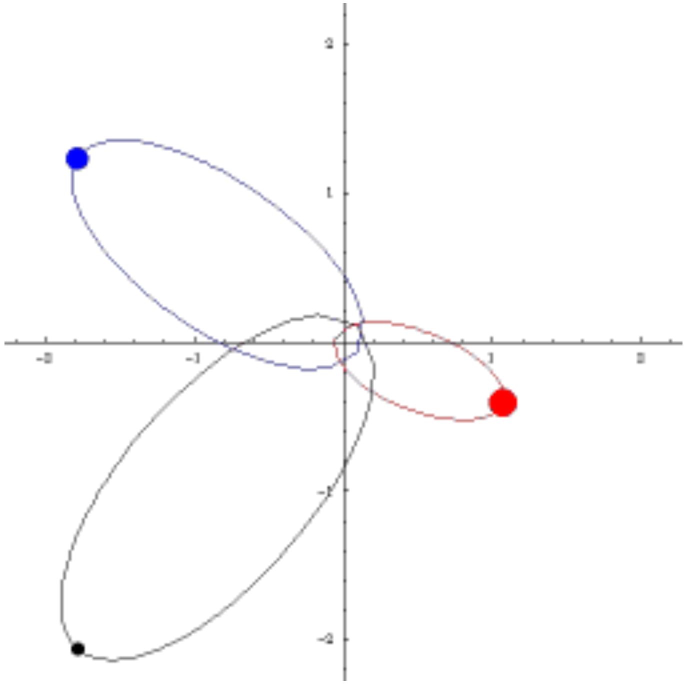
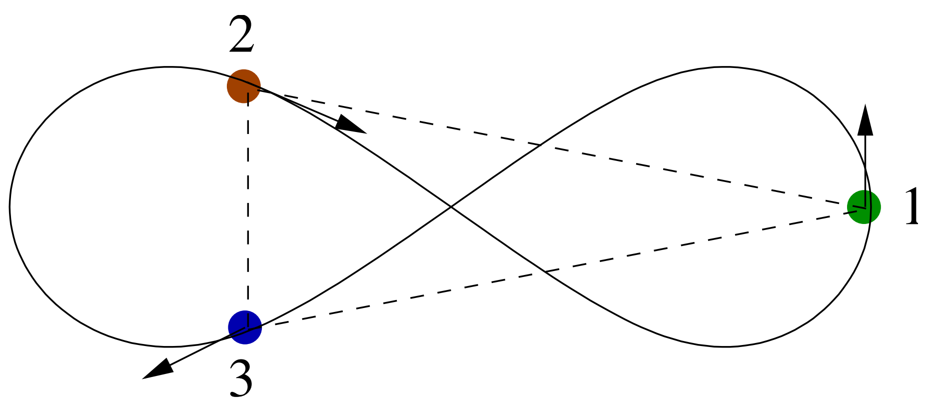


Figure-eight solution

Chenciner and Montgomery(Ann. of Math. 2000) proved the existence of an eight-shaped periodic solution in the planar three-body problem with equal masses by using the variationa method.

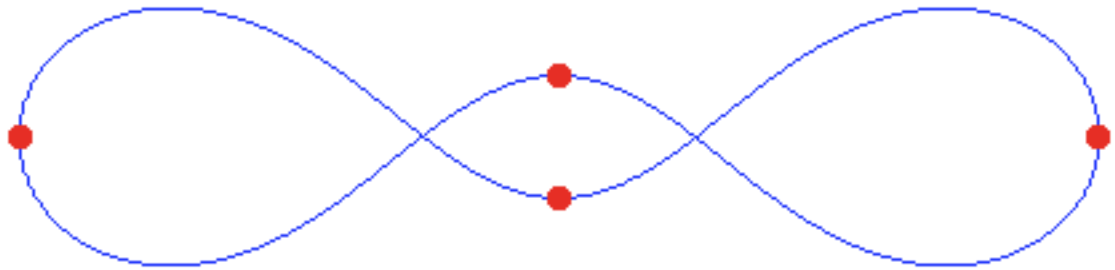


Gerver's super-eight orbit

Just after the discovery of the figure eight orbit, Gerver numerically found a similar periodic orbit in the planar four-body problem with equal masses.

In 2003 Kapela and Zgliczynski provided a computer-assisted proof for the existence. But there has been no variational proof yet.

The goal of this talk is to provide the variational existence proof for the super-eight orbit.



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P_x: $\mathbb{R}^2 \to \mathbb{R}$, the projection to x-axis.

P_y: $\mathbb{R}^2 \to \mathbb{R}$, the projection to y-axis.

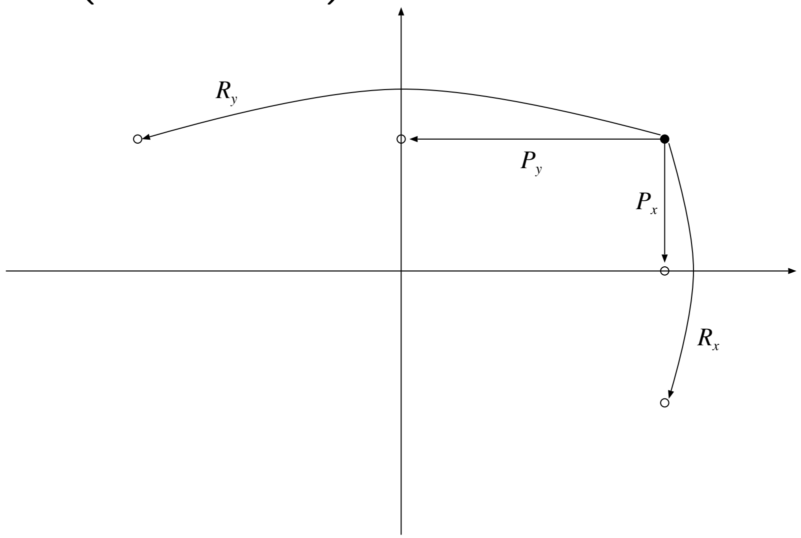
R_x: $\mathbb{R}^2 \to \mathbb{R}^2$, the reflection with respect to x-axis.

R_y: $\mathbb{R}^2 \to \mathbb{R}^2$, the reflection with respect to y-axis.

These are represented by the matrices

$$P_x = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad P_y = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

$$R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$



Consider the planar four-body problem with equal masses:

$$d = 2, N = 4, m_1 = m_2 = m_3 = m_4 = 1.$$

Main Theorem

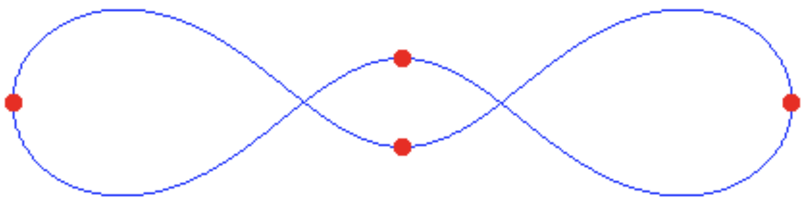
There is a collisionless 2π*-periodic solution* $(q_1(t), q_2(t), q_3(t), q_4(t)) : \mathbb{R} \to (\mathbb{R}^2)^4$ *of the planar four-body problem with equal masses such that for any* $t \in \mathbb{R}$

$$q_1(t) = R_y q_1(-t) = R_x q_2\left(\frac{\pi}{2} - t\right), \qquad q_2(t) = R_x q_2(-t) \qquad (1)$$

$$q_1(t) = -q_3(t), \qquad q_2(t) = -q_4(t) \qquad (2)$$

and that

$$P_y q_1(0) > 0, \qquad P_x q_2(0) > 0, \qquad P_x q_1\left(\frac{\pi}{4}\right) > 0, \qquad P_y q_1\left(\frac{\pi}{4}\right) < 0.$$



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Step 1(Existence of a generalized solution)

Consider the subsystem whose configuration holds $q_1 = -q_3, q_2 = -q_4$.

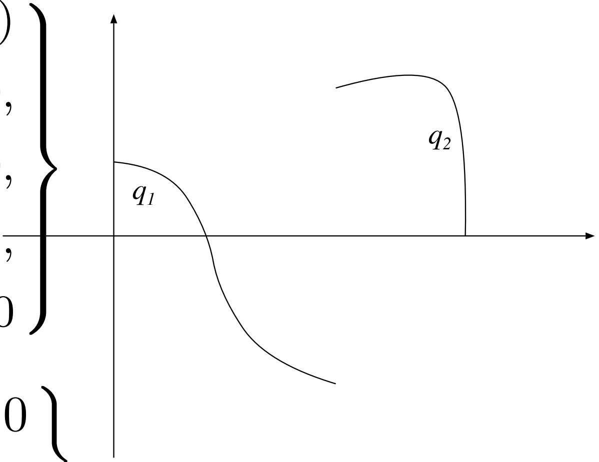
The action functional for the subsystem is

$$\mathcal{J}(\gamma) = \int_{T_0}^{T_1} \frac{1}{2}(|\dot{q}_1|^2 + |\dot{q}_2|^2) + \frac{1}{4|q_1|} + \frac{1}{4|q_2|} + \frac{1}{|q_1 - q_2|} + \frac{1}{|q_1 + q_2|} dt$$

where $\gamma(t) = (q_1(t), q_2(t)) : [T_0, T_1] \to (\mathbb{R}^2)^2$.

The domain of $\mathcal{J}$ is $\hat{\Gamma}$:

$$\Gamma = \left\{ \gamma(t) = (q_1(t), q_2(t)) \;\middle|\; \begin{array}{r} \gamma \in H^1([0, \pi/4], (\mathbb{R}^2)^2) \\ P_x q_1(0) = P_y q_2(0) = 0, \\ P_y q_1(0) \geq 0, \quad P_x q_2(0) \geq 0, \\ R_x q_1(\pi/4) = q_2(\pi/4), \\ P_x q_1(\pi/4) \geq 0, \quad P_y q_1(\pi/4) \leq 0 \end{array} \right\}$$



$$\hat{\Gamma} = \left\{ \gamma(t) = (q_1(t), q_2(t)) \in \Gamma \;\middle|\; \begin{array}{r} q_1(t) \neq 0, q_2(t) \neq 0 \\ q_1(t) \neq q_2(t), q_1(t) \neq -q_2(t) \end{array} \right\}.$$

1. Consider an action functional with a strong force part:

$$\mathcal{J}^{\varepsilon}(\gamma) = \mathcal{J}(\gamma) + \varepsilon \int_0^{2\pi} \frac{1}{4|q_1|^2} + \frac{1}{4|q_2|^2} + \frac{1}{|q_1 - q_2|^2} + \frac{1}{|q_1 + q_2|^2} dt,$$

   where $\gamma(t) = (q_1(t), q_2(t)) \in \hat{\Gamma}$.
2. There exists a collision-less minimizer $\gamma^{\varepsilon}(t) = (q_1^{\varepsilon}(t), q_2^{\varepsilon}(t))$ of $\mathcal{J}^{\varepsilon}$ in $\hat{\Gamma}$ for $\varepsilon > 0$.
3. Take a convergent subsequence $\gamma^{\varepsilon_n} \to \gamma^0 \in \Gamma, \varepsilon_n \to +0 (n \to \infty)$.
4. The limit $\gamma^0(t) = (q_1^0(t), q_2^0(t))$ is a generalized solution (which may have some collisions).
5. γ^0 is a minimizer of $\mathcal{J}$ in Γ.

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Existing methods for eliminating a collision

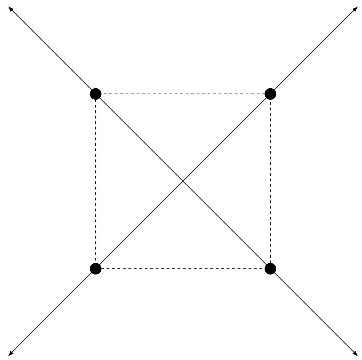
- Global estimate: estimate the lower bound of the action functional for any collision path and make a test path with lower value. (ex. Chenciner-Mongomery's proof for figure-eight 2000, Chen's 4-body orbit 2001)
- Local estimate (1): investigate the asymptotic behavior around the collision time with Sundman's estimate and modify the path around the collision time so that its value of the action functional becomes lower. (Chen's orbits with free boundary 2003, Venturelli's proof for Schubart orbit 2008, Shibayama's proof for Schubart-like orbits 2011)
- Local estimate (2):compute the average of the value of the action functional for modified paths in all direction near the collision time and prove that the average is lower the value than the original collision path(Marchal theorem 2002, Ferrario-Terracini theorem 2004)

In order to eliminate the collisions for the super-eight, a new technique is necessary.

Exclusion of a total collision(Global estimate)

Lemma 1. *If $\gamma \in \Gamma$ has a total collision,* $\mathcal{J}(\gamma) > 9$

Proof. The minimizer on $\{\gamma \in H^1([0,\pi/4],(\mathbb{R}^2)^2) \mid \gamma(t) = 0 \text{ for some } t \in [0,\pi/4]\}$ is the square collision-ejection orbit. From Gordon [1], the value of the action functional is $2^{-4/3} \cdot 3(1+2\sqrt{2})^{2/3}\pi = 9.15330757956509\cdots$. □

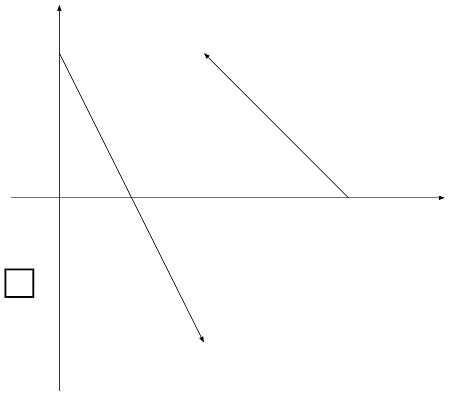


Lemma 2. $\inf_{\gamma\in\Gamma} \mathcal{J}(\gamma) < 5$

Proof. We take a test path $\gamma_{\text{test}} = (t, \frac{\pi}{4} - 2t, \frac{\pi}{2} - t, t)$.
From an easy computation,

$$\mathcal{J}(\gamma_{\text{test}}) \le \frac{7\pi}{8} + \frac{\sqrt{5}}{4} + \frac{\sqrt{2}}{8} + \frac{\sqrt{13}}{4} + \frac{1}{2} = 4.88607508042865\cdots$$

Therefore $\inf_{\gamma\in\Gamma} \mathcal{J}(\gamma) \le \mathcal{J}(\gamma_{\text{test}}) < 5$. □



Therefore γ^0 must not have a total collision since it is a minimizer.

[1] W. B. Gordon, A Minimizing Property of Keplerian Orbits, American Journal of Mathematics **99**, 961–971 (1977)

Exclusion of a binary collision

In order to eliminate binary collisions, we use Tanaka's scaling technique:

The case of $q_1^0(0) = 0$.

We use Tanaka's scaling technique [2].

Define a scale transformation x_n of $q_1^{\varepsilon_n}$ by

$$x_n(s) = \delta_n^{-1} q_1^{\varepsilon_n}(\delta_n^{3/2} s) \qquad \delta_n = |q_1^{\varepsilon_n}(0)|.$$

By taking a subsequence of n if necessary, we can assume

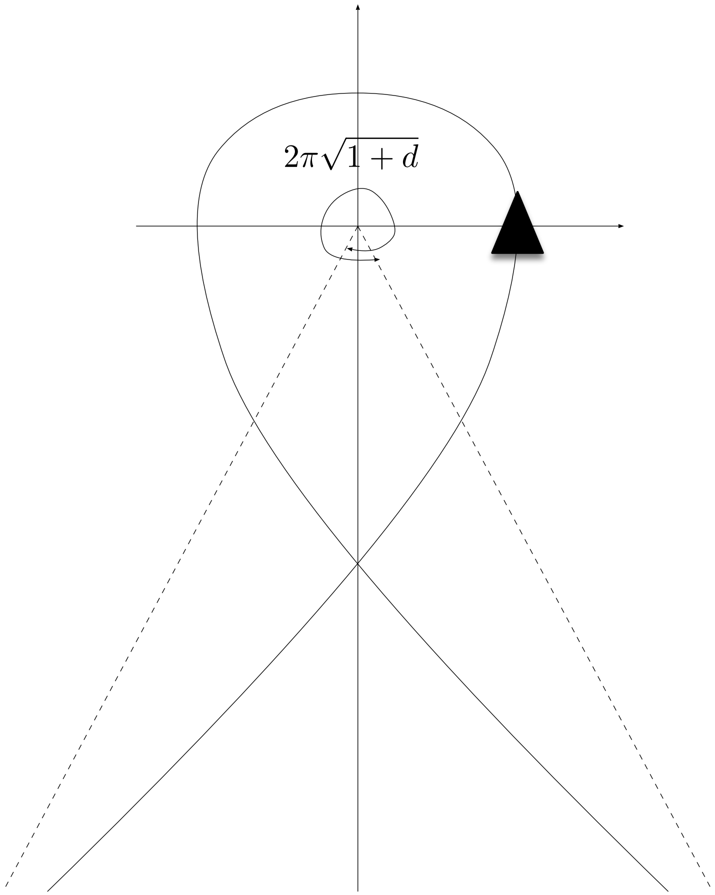
$$\frac{\varepsilon_n}{\delta_n} \to d \in [0, \infty].$$

We consider the case of $d < \infty$. Fix any $l > 0$. x_n converges uniformly on $[0, l]$ to the solution $y_d(s)$ of

$$\frac{d^2 y}{ds^2} + \frac{y}{4|y|^3} + d\frac{y}{|y|^4} = 0, \quad y(0) = (0, 1), \dot{y}(0) = \left(\pm\sqrt{\frac{1}{2} + 2d}, 0\right).$$

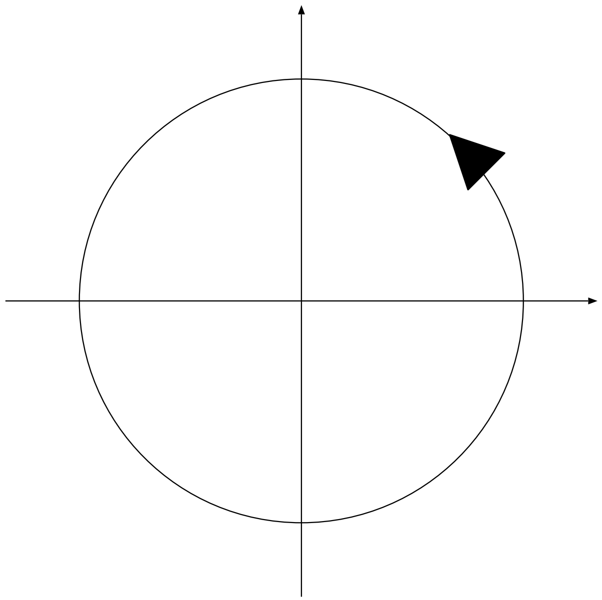
[2] K. Tanaka, Noncollision solutions for a second order singular Hamiltonian system with weak force. Ann. Inst. H. Poincaré Anal. Non Linéaire **10**, 215–238 (1993)

d<∞



We can understand the behavior of $y_d(s)$ well.

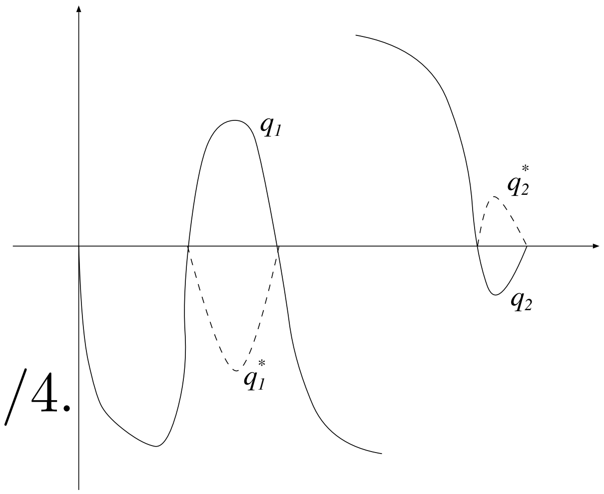
d=∞



In the case of $d=\infty$, we use another scaling coordinates.

In the case of $d = 0$, Tanaka [3] showed

$$\lim_{t \to +0} \frac{q_1(t)}{|q_1(t)|} = (0, -1).$$



If $\dot{q}_2(0) = 0$, the total collision occurs at $t = \pi/4$.

If $P_y \dot{q}_2(0) < 0$, $q_1(t)$ and $q_2(t)$ moves into forth quadrant for small $t > 0$.

If $P_y \dot{q}_2(0) > 0$, $q_1(t)$ and $q_4(t)$ moves into third quadrant for small $t > 0$.

We can reflect the curve such that each particle moves in the separate quadrants. The value of action functional is lower.

[3] K. Tanaka, A note on generalized solutions of singular Hamiltonian systems. Proc. Amer. Math. Soc. **122**, 275–284 (1994)

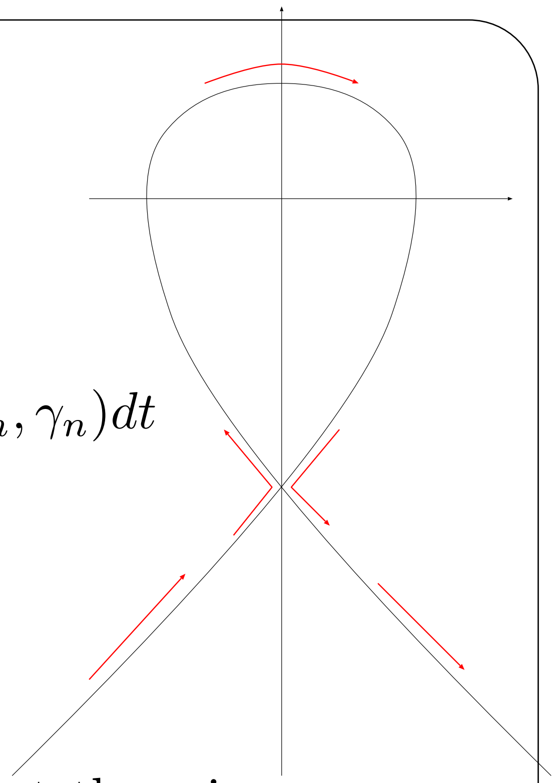
Consider the case of $d > 0$.

The action functional can be represented by

$$\mathcal{J}^{\varepsilon_n}(\gamma_n) = \delta_n^{1/2}\mathcal{I}_l^{\varepsilon_n/\delta_n}(x_n) + O(\delta_n^{3/2}\varepsilon_n) + \int_{\delta_n^{3/2}l}^{\pi/4} L^{\varepsilon}(\dot{\gamma}_n, \gamma_n)dt$$

where

$$\mathcal{I}_l^a(x) = \int_0^l \frac{1}{2}|\dot{x}(s)|^2 + \frac{1}{4|x(s)|} + \frac{a}{|x(s)|^2}ds.$$



$y_d(s) = \lim_{n\to\infty} x_n(s)(s \in [0,l])$ is a minimizer of $\mathcal{I}_l^d$. But the minimizer can not have self-intersection(whose proof is similar to Coti-Zerati's idea [4]). This is a contradiction.

The case of $d = \infty$ can be eliminated similarly since the path must have self-intersection.

[4] V. Coti Zelati, Periodic solutions for a class of planar, singular dynamical systems, *J. Math. Pures Appl.* **68** (1989), 109–119.

- The binary collision $q_2(0) = 0$ can be eliminated similarly.

- The binary collision at $t \in (0, \pi/4)$ can be eliminated by applying Ferrario-Terracini's theorem[5].

- The binary collision at $t = \pi/4$ can be reduced to the case of $t = 0$, since the situation is essentially same. In fact, the action functional can be written by

$$\mathcal{J}(\gamma) = \int_0^{\pi/4} \frac{1}{2}(|\dot{Q}_1|^2 + |\dot{Q}_2|^2) + \frac{1}{\sqrt{2}|Q_1|} + \frac{1}{\sqrt{2}|Q_2|} + \frac{1}{\sqrt{2}|Q_1 - Q_2|} + \frac{1}{\sqrt{2}|Q_1 + Q_2|} dt$$

where

$$Q_1 = \frac{q_1 + q_2}{\sqrt{2}}, Q_2 = \frac{q_1 - q_2}{\sqrt{2}}.$$

The boundary condition at $t = \pi/4$ is

$$P_x Q_1(\pi/4) = P_y Q_2(\pi/4) = 0, P_y Q_1(\pi/4) > 0, P_x Q_2(\pi/4) < 0.$$

[5] D. L. Ferrario & S. Terracini, On the existence of collisionless equivariant minimizers for the classical n-body problem, *Invent. Math.* **155**, 305–362 (2004)

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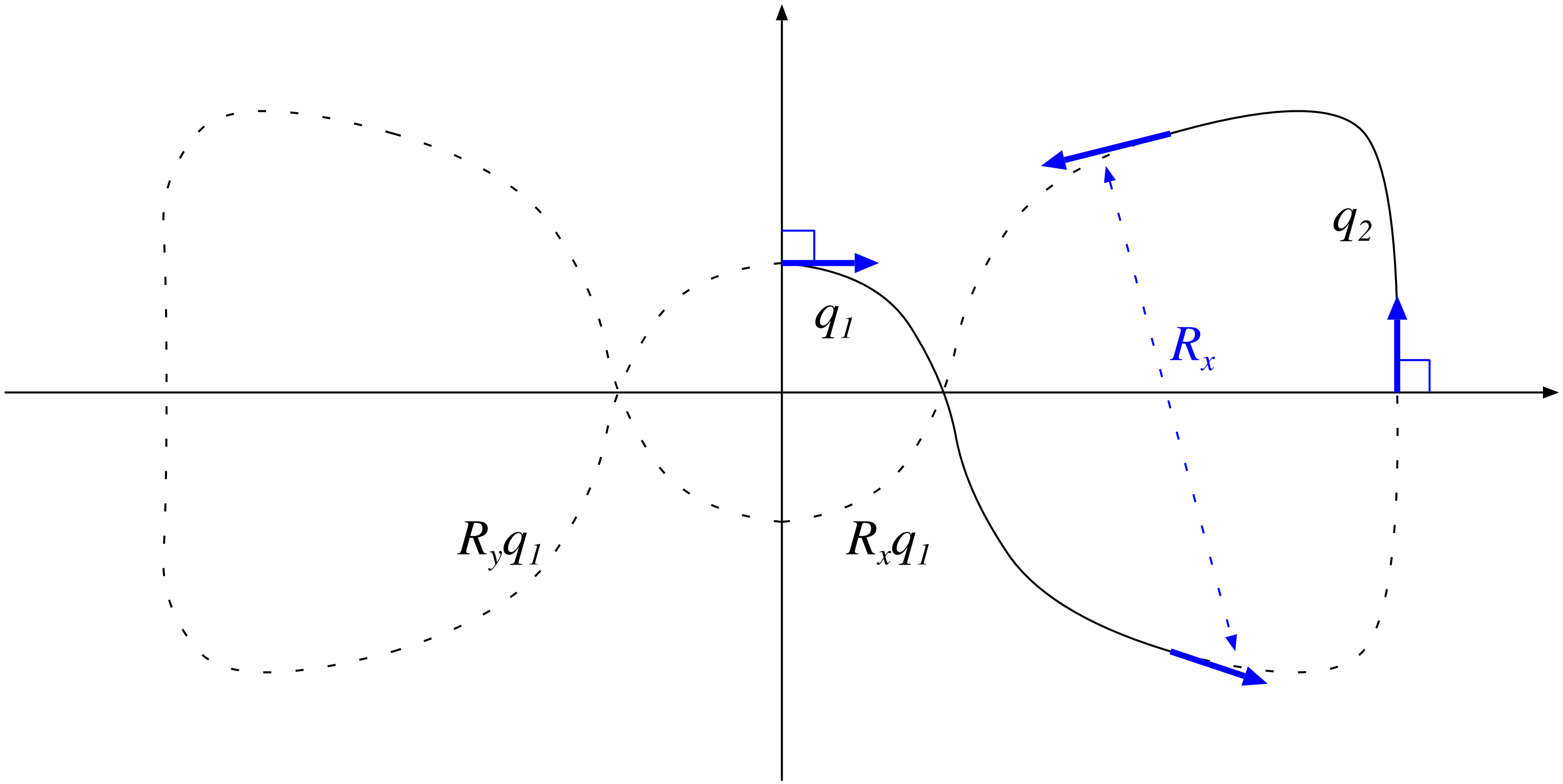
Symmetry

The minimizer can be smoothly connected with the reflected paths smoothly.



This completed the proof !

Summary

- We prove the existence of the super-eight by using variational method.
- The most difficult part is to eliminate the possibility of collisions.
- In order to solve the difficulty, we apply Tanaka's technique.

Thank you for your attention!