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on Celestial Mechanics and N-body Dynamics
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General Relativistic Three-body Problem

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Main references on Hirosaki papers

Chiba, Imai, HA,
Mon. Not. Roy. Astr. S, 377, 269 (2007) [ArXiv:astro-ph/0609773]

Imai, Chiba, HA,
Phys. Rev. Lett. 98, 201102 (2007) [ArXiv:gr-qc/0702076]

Torigoe, Hattori, HA,
Phys. Rev. Lett. 102, 251101 (2009) [ArXiv:gr-qc/0906.1448]

HA,
Phys. Rev. D 80, 064021 (2009) [ArXiv:gr-qc/1010.2284]

Yamda, HA,
Phys. Rev. D 82, 104019 (2010) [ArXiv:gr-qc/1010.2284]

Yamda, HA,
Phys. Rev. D 83, 024040 (2011) [ArXiv:gr-qc/1011.2007]

Ichita, Yamda, HA,
Phys. Rev. D 83, 084026 (2011) [ArXiv:gr-qc/1011.3886]

Yamda, HA,
Phys. Rev. D 86, 124029 (2012) [Arxiv:gr-qc/1212.0754]

Yamda, HA,
Mon. Not. Roy. Astr. S, 423, 3540 (2012) [Arxiv:gr-qc/1204.5298]

N-body Problem

in Newton gravity

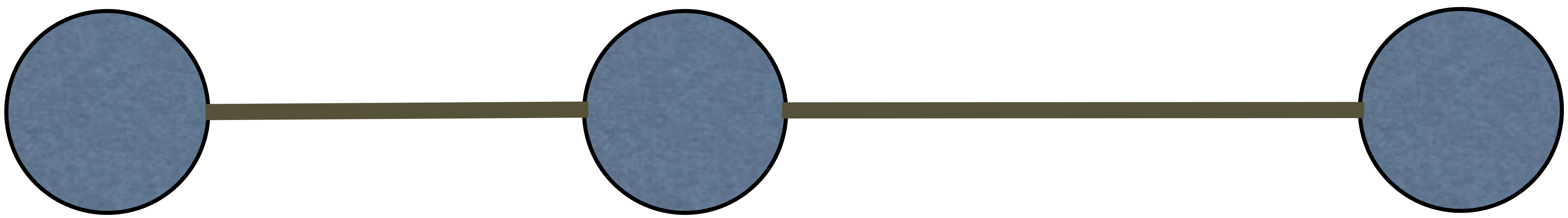
2-body problem

solved by (E, L)

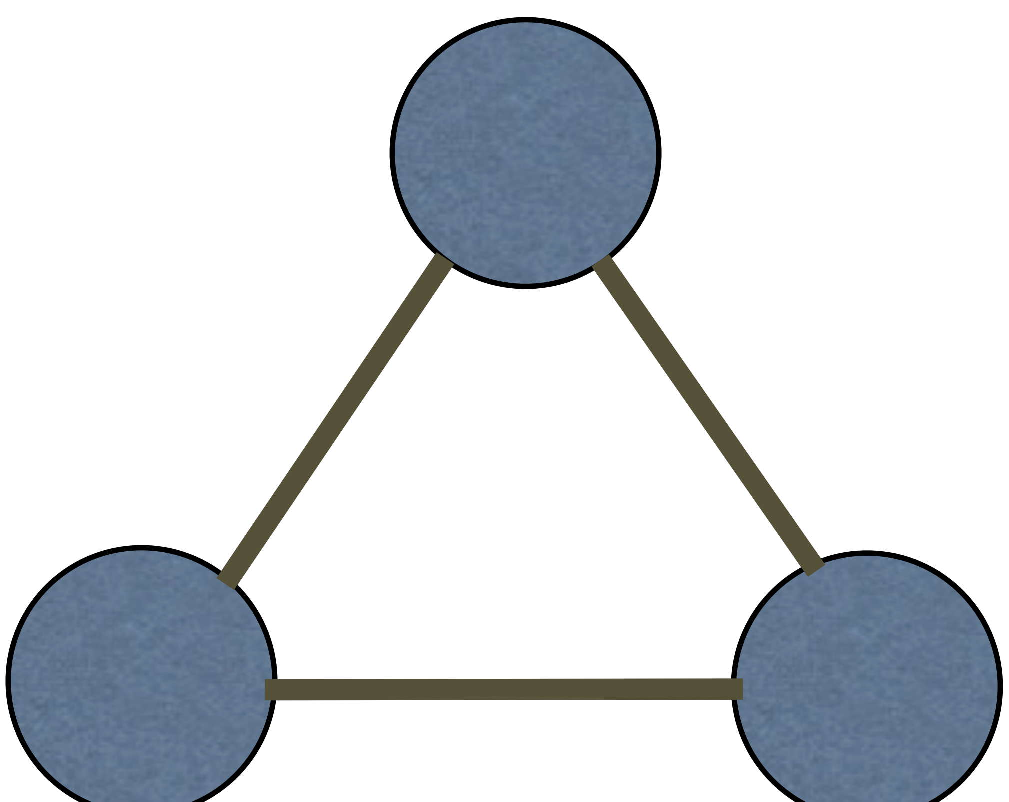
elliptic	$E < 0$
parabolic	$E = 0$
hyperbolic	$E > 0$

3-body

Euler's collinear solution (1765)



Lagrange's triangle (1772)



Henri Poincare

N = 3 (or more)

impossible to describe all the solutions to the N-body problem.

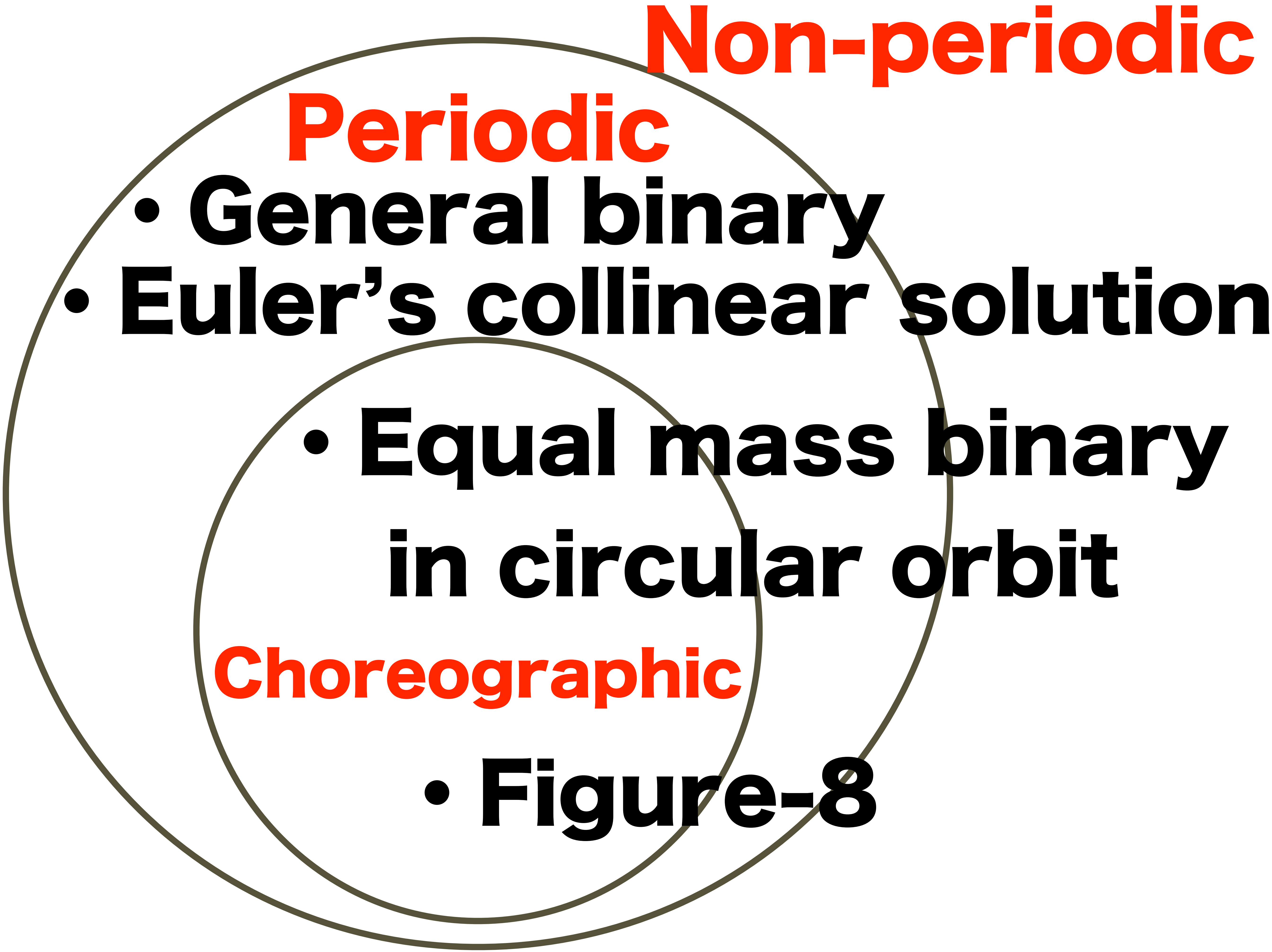
# of new solutions is increasing.

Remarkable one was found:

Figure-eight solution!

Moore,
Phys. Rev. Lett. 70, 3675 (1993)

Chenciner, Montgomery,
Ann. Math. 152, 881 (2000)


Non-periodic
Periodic
• General binary
• Euler’s collinear solution
• Equal mass binary
in circular orbit
Choreographic
• Figure-8

Let us re-examine

3-body problem

in the framework of

general relativity

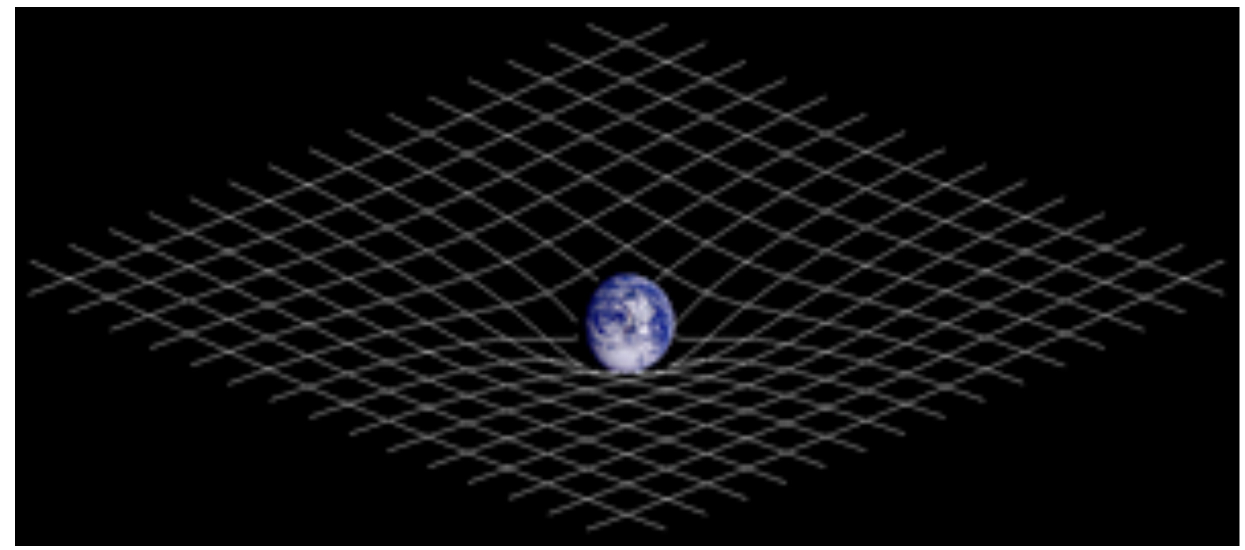
(Einstein gravity)

GR = General Relativity

Newton

Gravity = Force

Einstein

Gravity = 

Curved Space-time

↓

light ray bends

gravitational waves

$$G_{\mu\nu} = T_{\mu\nu}$$

Space-time Geometry

Matter Energy -Momentum

Post-Newtonian approx.

Newton + 1PN + 2PN + ...

$$\left(\frac{v}{c}\right)^2 \qquad \left(\frac{v}{c}\right)^4$$

Dominant corrections

General relativistic effects

Periastron advance

Mercury

Time delay

GPS, Viking, Cassini

Light bending

Gravitational Lens

Binary pulser

Hulse-Taylor

GW=Gravitational Waves

Tiny ripples of a curved space-time

Generated by accelerated masses

No direct detection so far

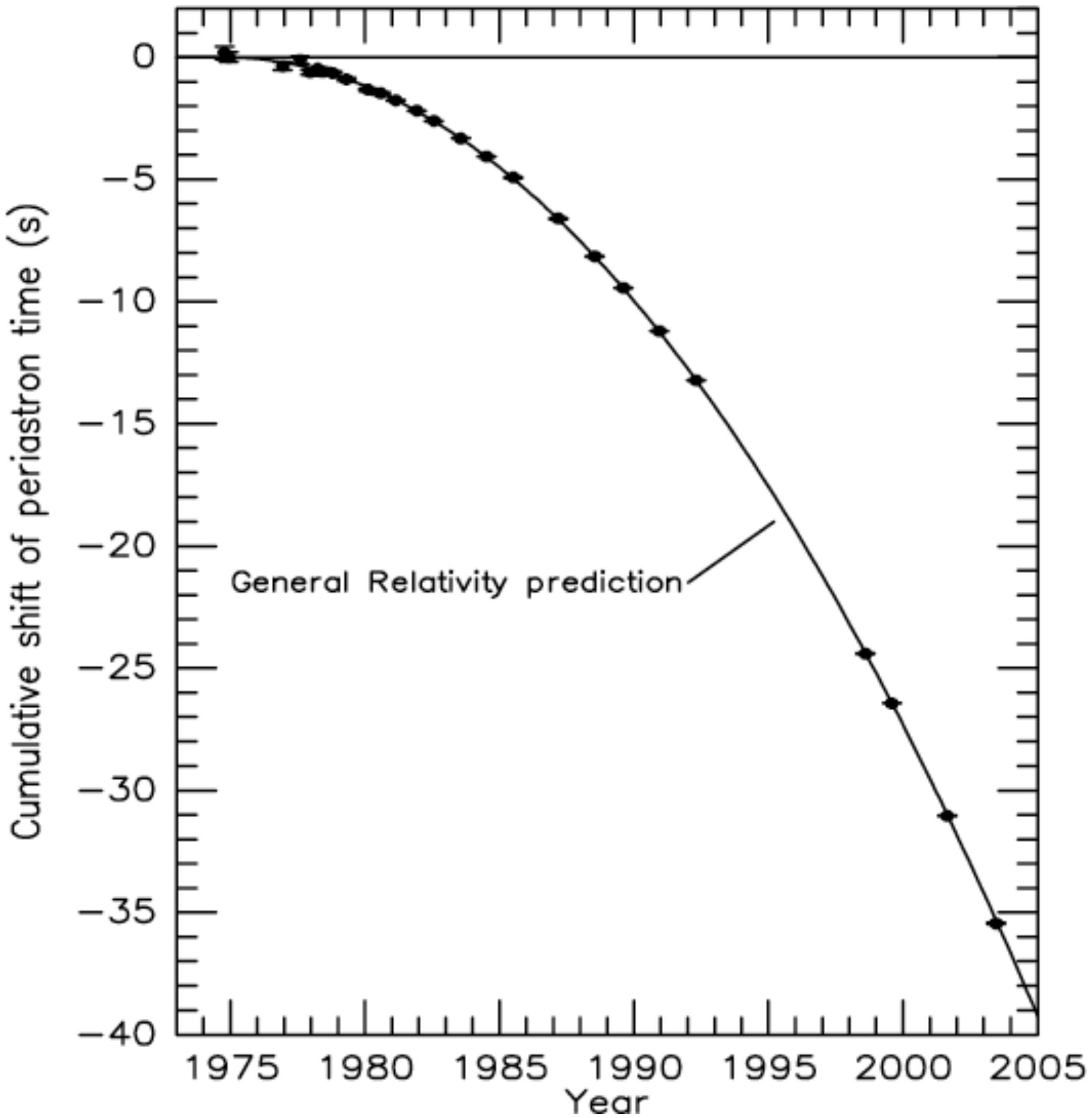
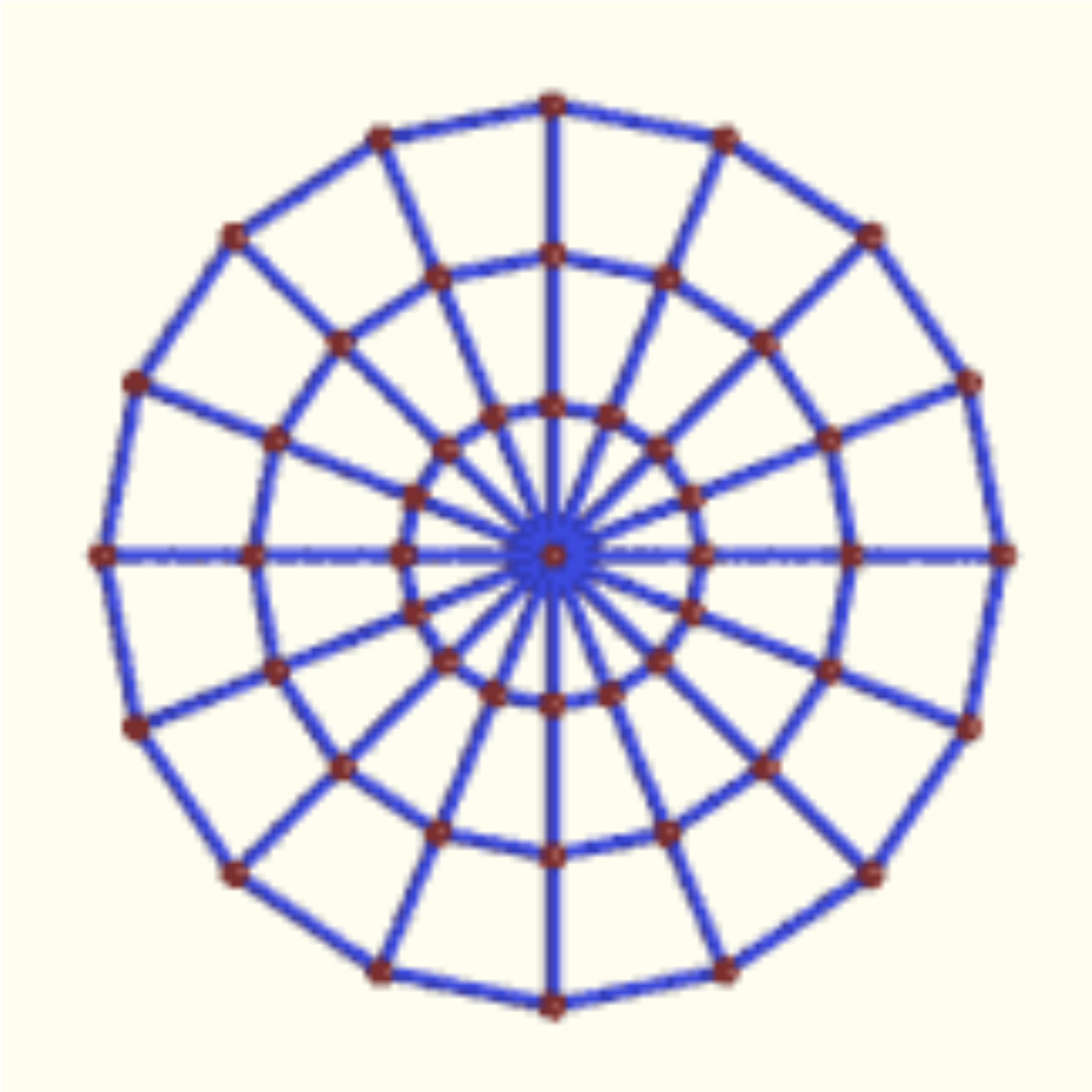


Figure 7: Plot of the cumulative shift of the periastron time from 1975 – 2005. The points are data, the curve is the GR prediction. The gap during the middle 1990s was caused by a closure of Arecibo for upgrading [272].

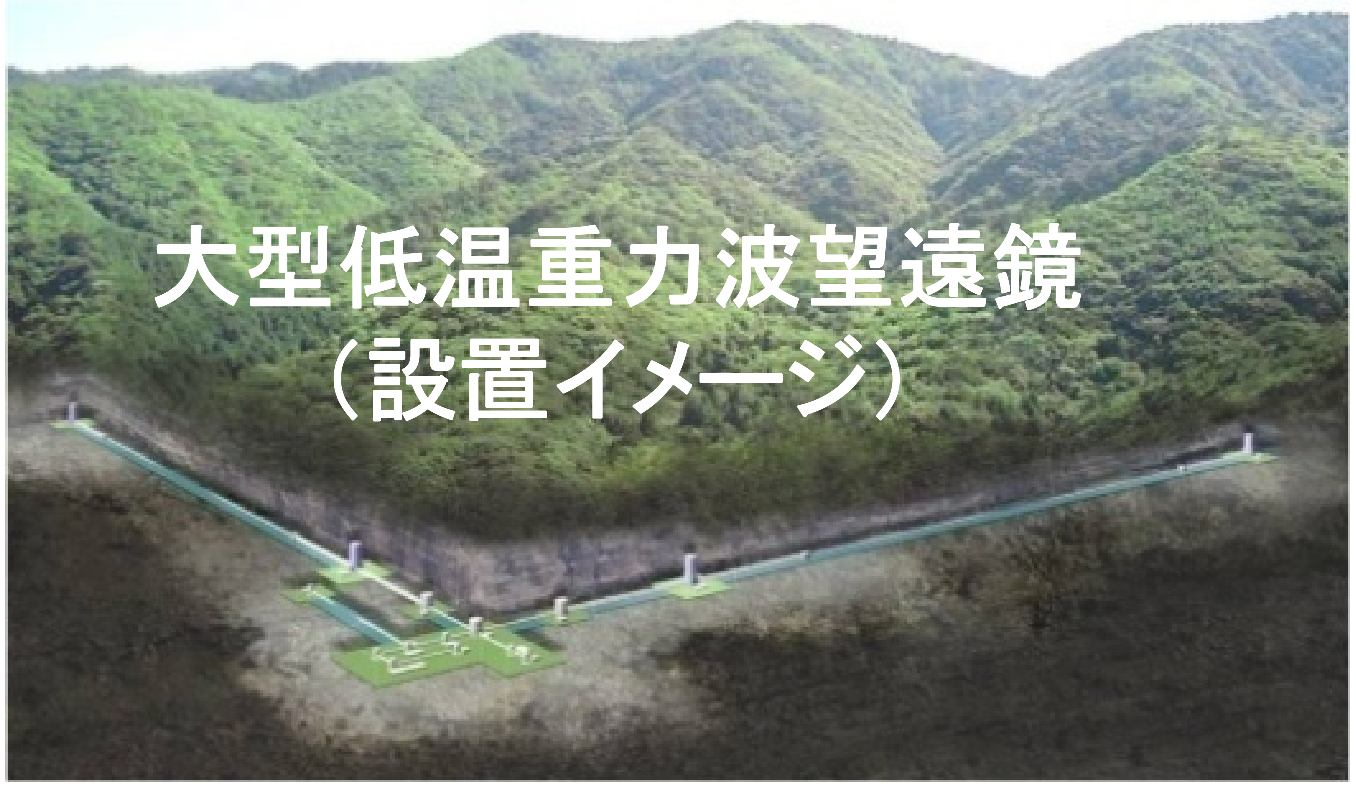
Will, LRR (06)

indirect evidence

by Binary Pulser



http://www.aei.mpg.de/einsteinOnline/en/spotlights/gw_waves/index.html



KAGRA(Japan)

VIRGO(Italy-France)

LIGO(US)

GEO600(UK-Germany)

Part 1: Choreography

Part 2: Euler+Lagrange's solutions

In Celestial Mechanics,

a solution is

'choreographic'

if

every massive particles

move periodically

in a single closed orbit

1)

Implication of Choreography to GR

2)

Effects of GR to Choreography

1)

Implication of Choreography to GR

2)

Effects of GR to Choreography

Promising GW sources

N=1

Rapidly Rotating Star

N=2

Compact Binary System

N=3 (or more)

much less attention

Because of Chaos

irregular waveform

↓

difficult to detect

Our question

Can three (or more) bodies generate period GW?

Ans.

Yes!

Chiba, Imai, HA,
Mon. Not. Roy. Astr. S, 377, 269 (2007)
Arxiv:astro-ph/0609773.

One example

Figure-8

Assumptions

The same plane

The same mass

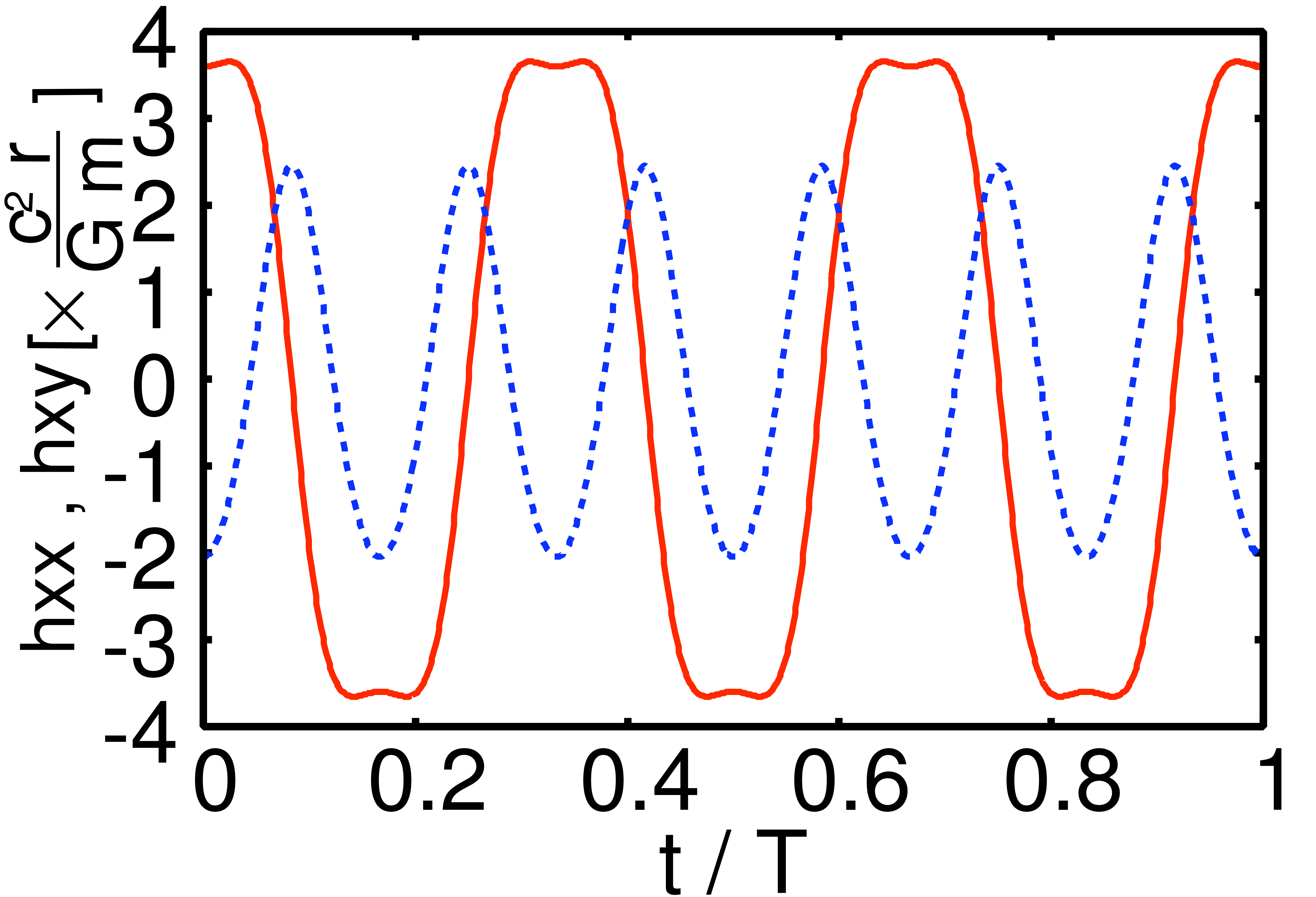
Computing Waveform

via Quadrupole formula

$$h_{ij}^{TT} = \frac{2G\ddot{Q}_{ij}}{rc^4} + O\left(\frac{1}{r^2}\right)$$

$$Q_{ij} = I_{ij} - \delta_{ij}\frac{I_{kk}}{3}$$

$$I_{ij} = \sum_{A=1}^{N} m_A x_A^i x_A^j$$


hxx , hxy [× c² r / G m]
4
3
2
1
0
-1
-2
-3
-4
0
0.2
0.4
0.6
0.8
1
t / T

1)
Implication of Choreography to GR

2)
Effects of GR to Choreography

2nd question

Newton's EOM is OK?

Ans.

No!

Imai, Chiba, HA,
Phys. Rev. Lett. 98, 201102 (2007)
Arxiv:gr-qc/0702076.

Einstein-Infeld-Hoffman Equation of motion

G = c = 1

$$\frac{d^2\boldsymbol{x}_K}{dt^2} = \sum_{A\neq K} \boldsymbol{r}_{AK}\frac{m_A}{r_{AK}^3}\Big[1-4\sum_{B\neq K}\frac{m_B}{r_{BK}}$$

$$-\sum_{C\neq A}\frac{m_C}{r_{CA}}\left(1-\frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)$$

$$+v_K^2+2v_A^2-4v_A\cdot v_K-\frac{3}{2}\left(\frac{\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK}}{r_{AK}}\right)^2\Big]$$

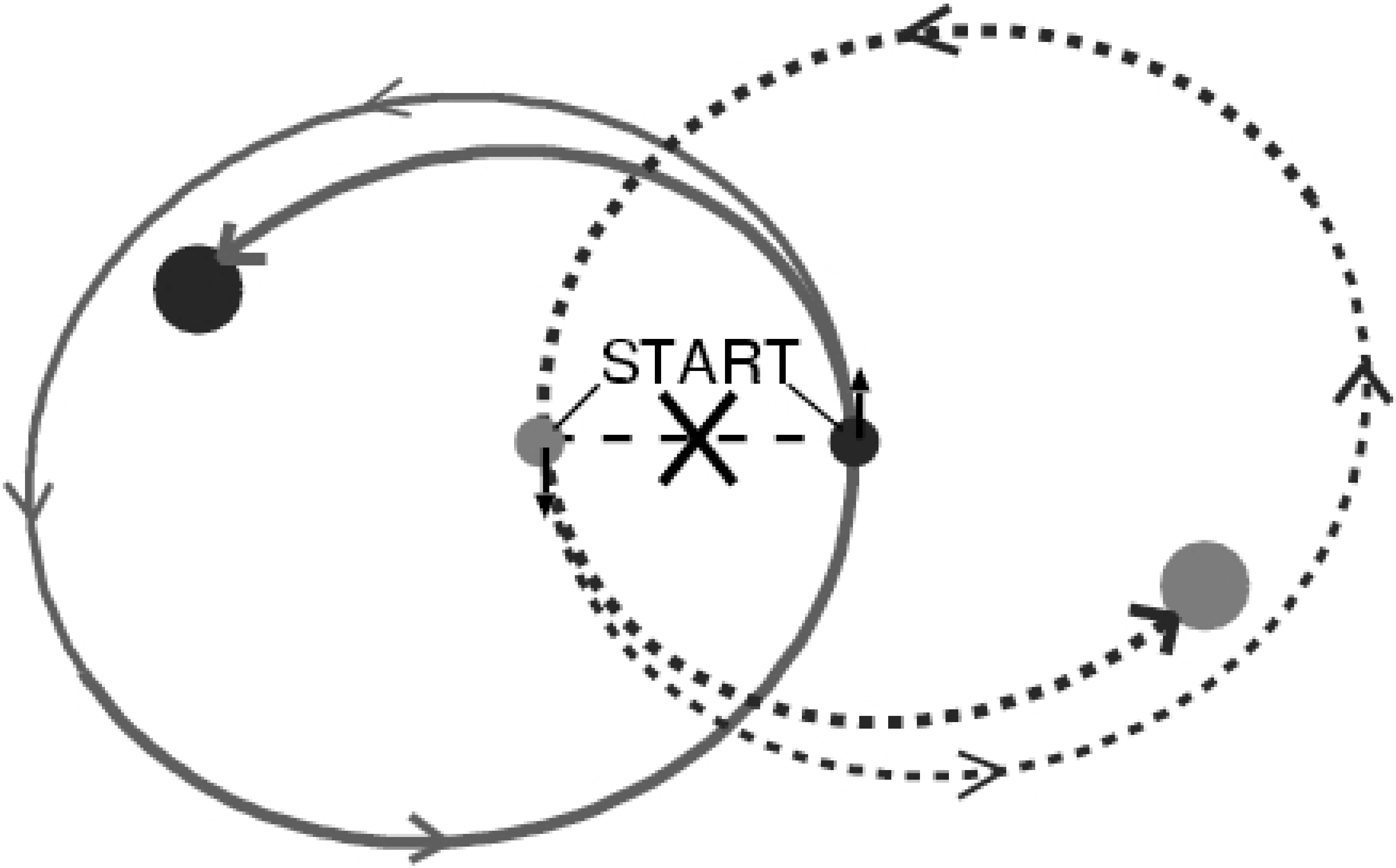
$$-\sum_{A\neq K}(\boldsymbol{v}_A-\boldsymbol{v}_K)\frac{m_A\boldsymbol{r}_{AK}\cdot(3\boldsymbol{v}_A-4\boldsymbol{v}_K)}{r_{AK}^3}$$

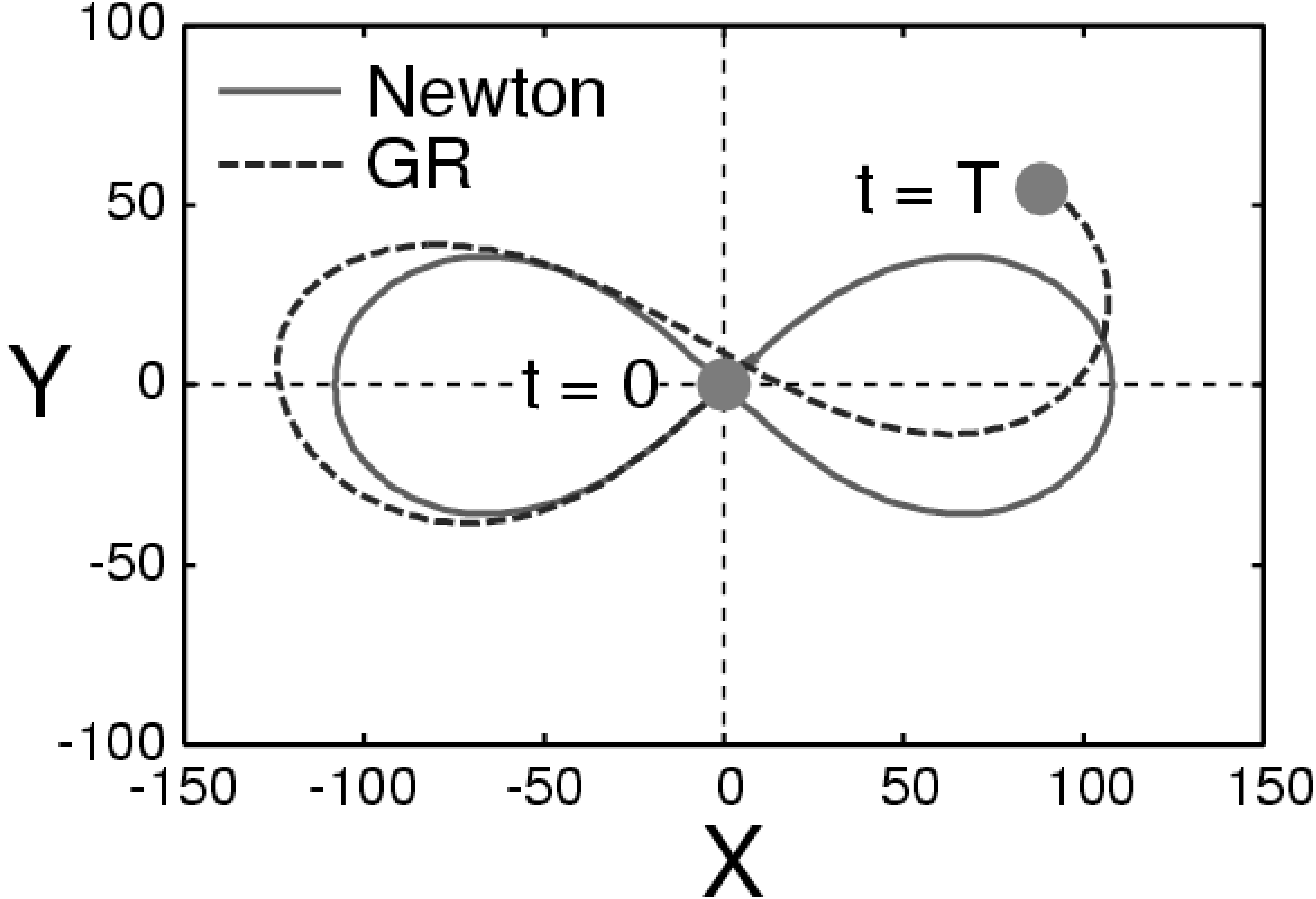
$$+\frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{m_Am_C}{r_{AK}r_{CA}^3}$$

A specific question:

For 2 bodies,
orbits cannot be closed
because of
periastron advance.

What happens
for figure-8 ?


START



Imai, Chiba, HA (2007)

Parametrise initial velocity

$$\vec{v}_1 = k\vec{V} + \xi\frac{m}{\ell^3}(\vec{V}\cdot\vec{\ell})\vec{\ell}$$

$$\vec{v}_2 = k\vec{V} + \xi\frac{m}{\ell^3}(\vec{V}\cdot\vec{\ell})\vec{\ell}$$

$$\vec{v}_3 = \vec{V}$$

$$k = -\frac{1}{2} + \alpha|\vec{V}|^2 + \beta\frac{m}{\ell}$$

$$\vec{P}_{tot} = \vec{L}_{tot} = 0$$

$$\alpha = -\frac{3}{16}$$

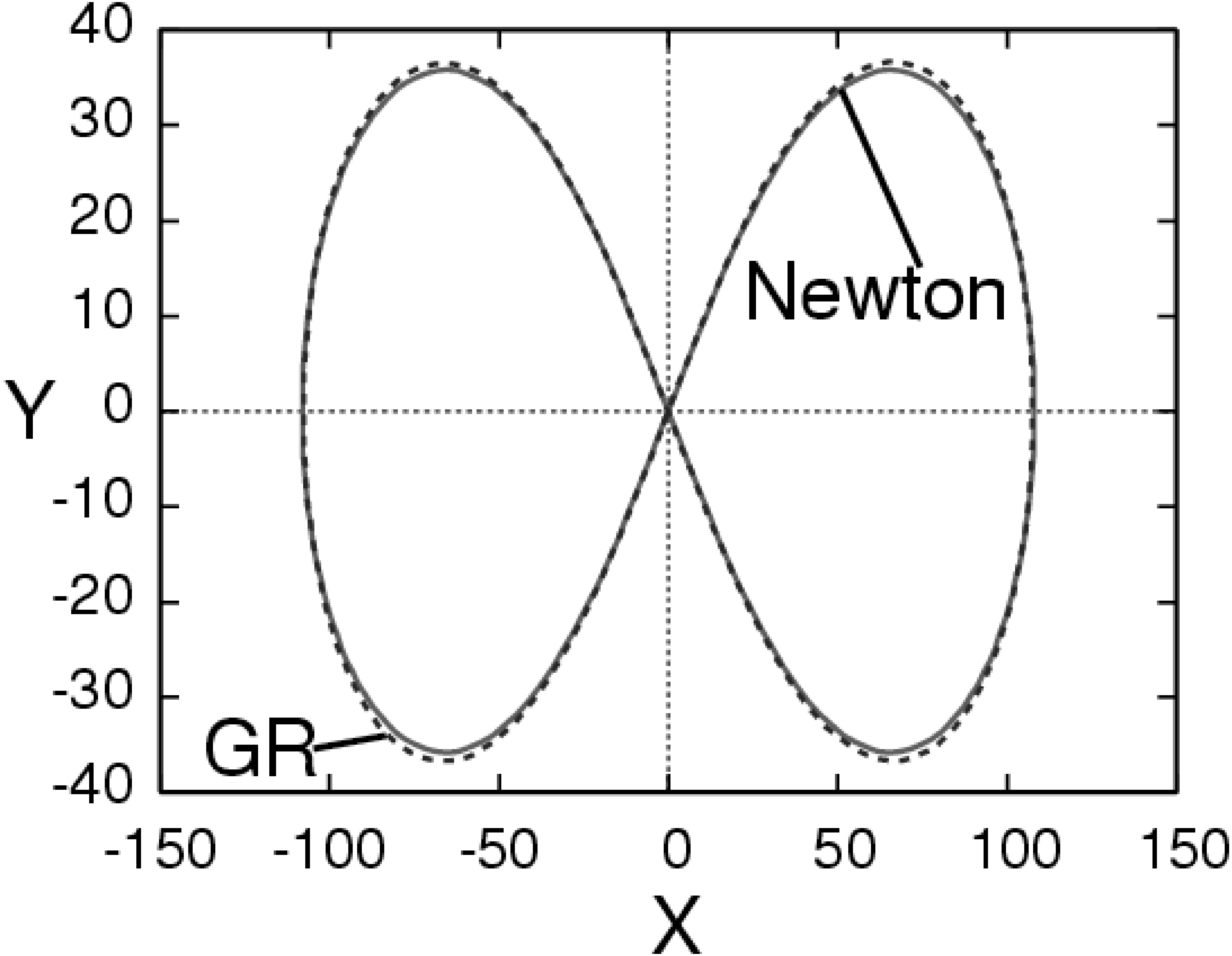
$$\beta = \xi = \frac{1}{8}$$

Remaining degrees of freedom

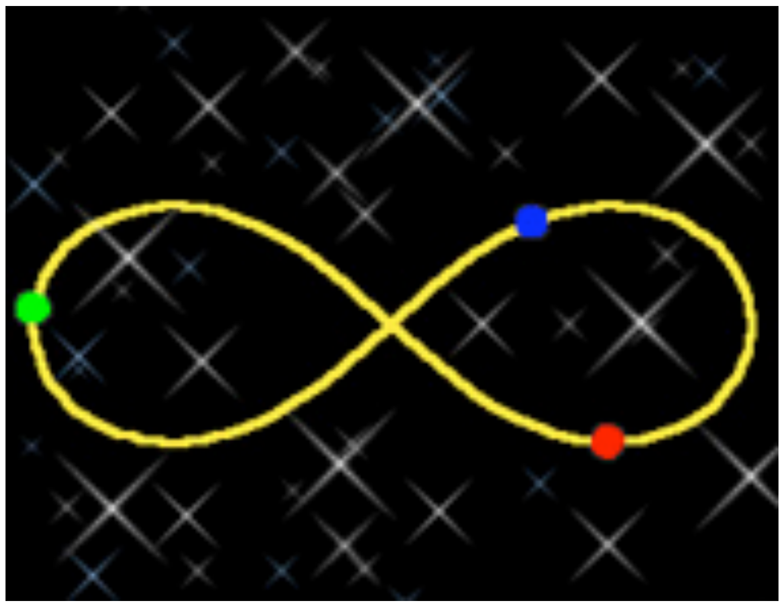
$$\vec{V} = (V_x, V_y)$$

are numerically determined.

(same as Newton figure-8)



Imai, Chiba, HA (2007)



On track.
A planetary figure-8 orbit is possible, at least temporarily, even if theorists account for the effects of general relativity.

Credit: Adapted from Michael Nauenberg / UC Santa Cruz

Trick Three-Planet Orbit Remains True

By Adrian Cho
*Science*NOW Daily News
4 May 2007

If a supreme being were so inclined, it could configure three planets so that they would race around one another in a graceful figure-8 orbit. At least that's what Newton's theory of gravity predicts. Now, a team of physicists has shown that the figure-8 orbit is possible even if they use Einstein's more-accurate theory of gravity, general relativity.

When two planets cling to each other through gravity, one will orbit the other by tracing an ellipse over and over. But throw together three or more orbs, and their interactions become so complex that chaos reigns. (Our solar system remains orderly because the sun is so heavy that each planet follows its lead and more or less ignores the other planets.) However, in 1993 physicist Cristopher Moore of the University of New Mexico in Albuquerque discovered that if he set things up just right, then according to Newton's theory, three equal-mass planets could chase each other endlessly in a figure 8.

It wasn't clear that Einstein's theory would allow the figure 8, however. General relativity says that gravity is actually the warping of space and time themselves, and it makes small but profound changes to the predictions of Newton's theory. For example, general relativity predicts that when one planet orbits another body, its orbit will slowly turn, like the hour hand on a clock, producing a complicated flowerlike pattern that doesn't repeat. In fact, once Einstein had completed the theory, he immediately showed that it could account for the theretofore unexplained turning of the orbit of Mercury. The figure-8 orbit ought to suffer from similar distortions.

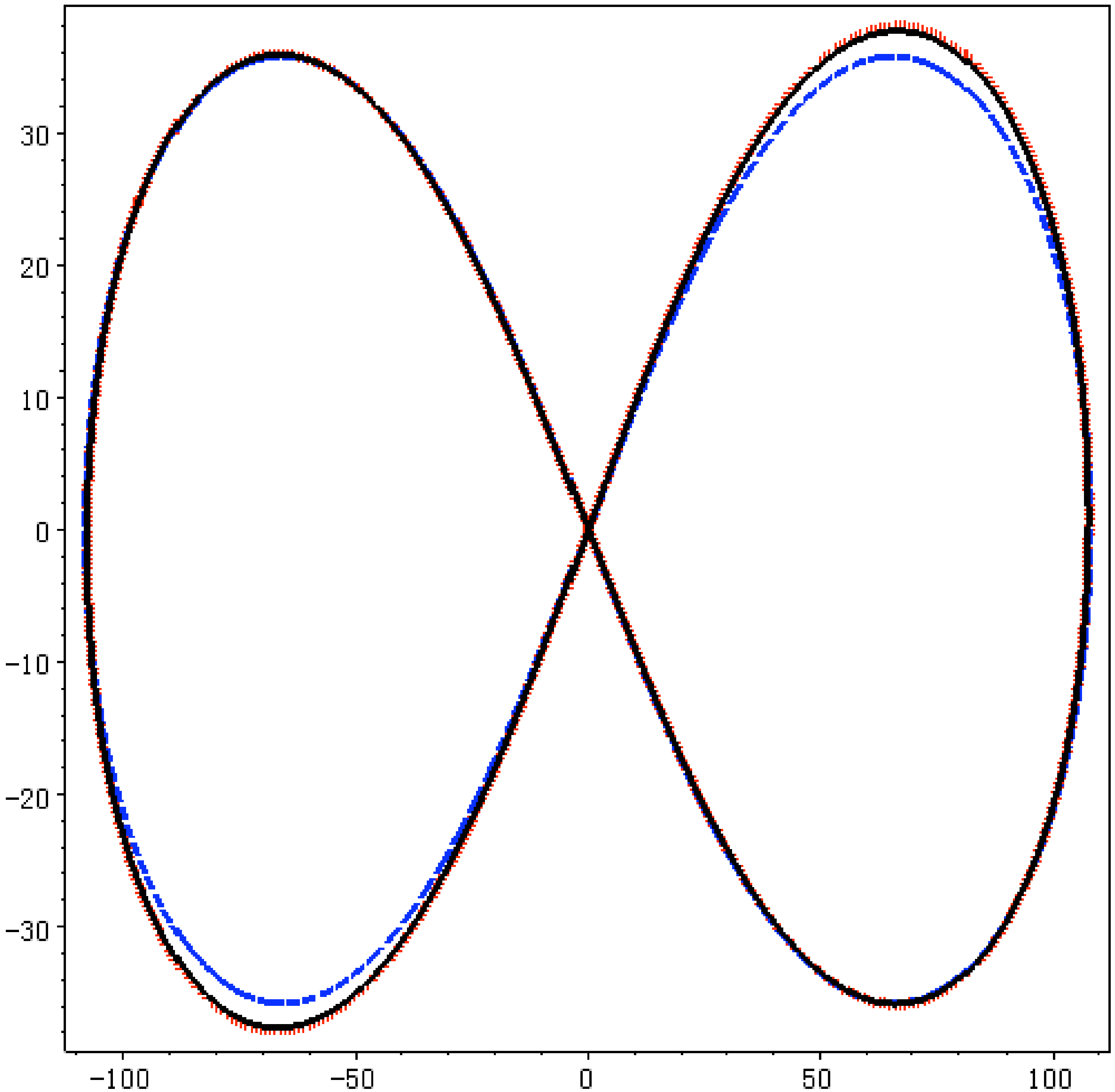
But Tatsunori Imai, Takamasa Chiba, and Hideki Asada at Hirosaki University in Japan have found that they can fiddle with the precise starting positions and velocities to compensate for the distortions and keep the planets on the figure-8 orbit, at least in the short term. Using a computer to simulate the exact orbit, they find that the planets stay on track for at least 10 cycles, as they report in an upcoming issue of *Physical Review Letters*. The analysis is the first to show that such an oddball orbit is possible in Einstein's universe.

"This is indeed an interesting and amazing result," says Luc Blanchet, a theoretical physicist at the Institute of Astrophysics in Paris. He notes, however, that in its full glory, Einstein's theory says the circulating planets should also produce ripples in space and time that will gradually carry away the planets' energy. That will eventually spoil the repeating orbit, Blanchet predicts: "I don't expect the figure 8 to remain [indefinitely]."

Trick Three-Planet Orbit Remains True

By Adrian Cho
*Science*NOW Daily News
4 May 2007

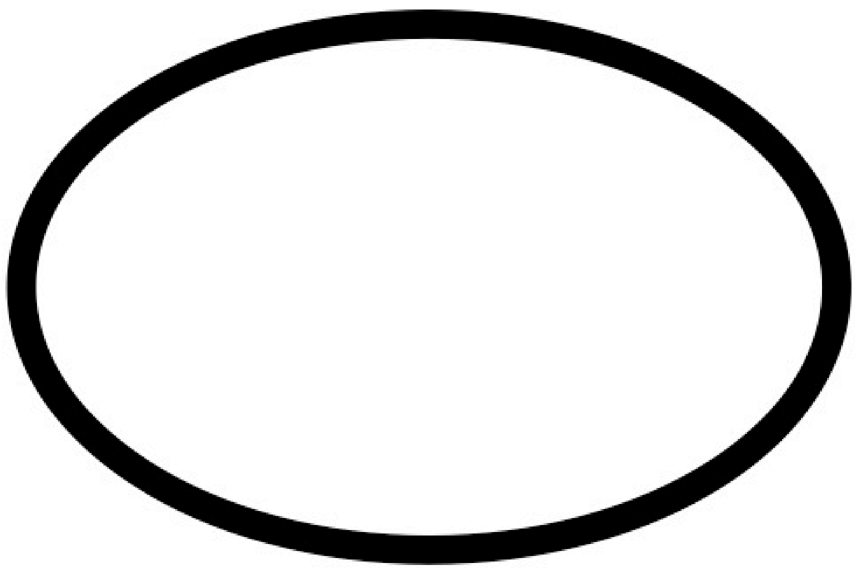
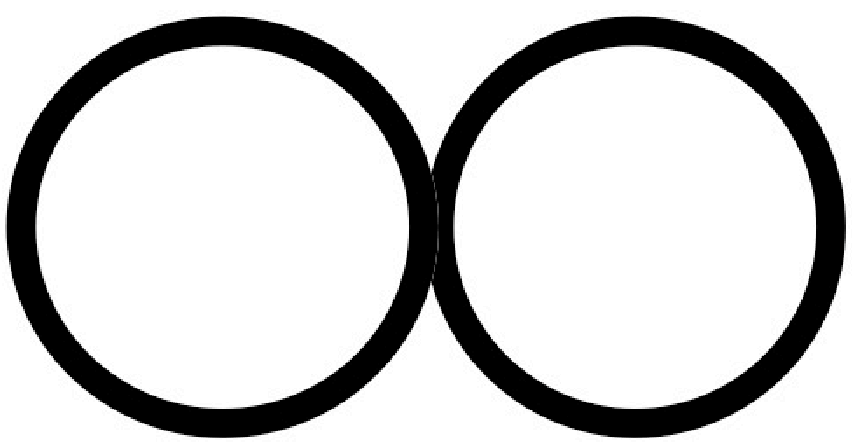
An extension to 2PN $\left(\frac{v}{c}\right)^4$



Lousto, Nakano, Class. Q. Grav. 25, 195019 (2008)

FIG. 9: Comparison of figure-eight motions for $\lambda = 1$. The solid, dotted and dashed lines show the 2PN, 1PN and Newtonian results, respectively.

Choreography or Not

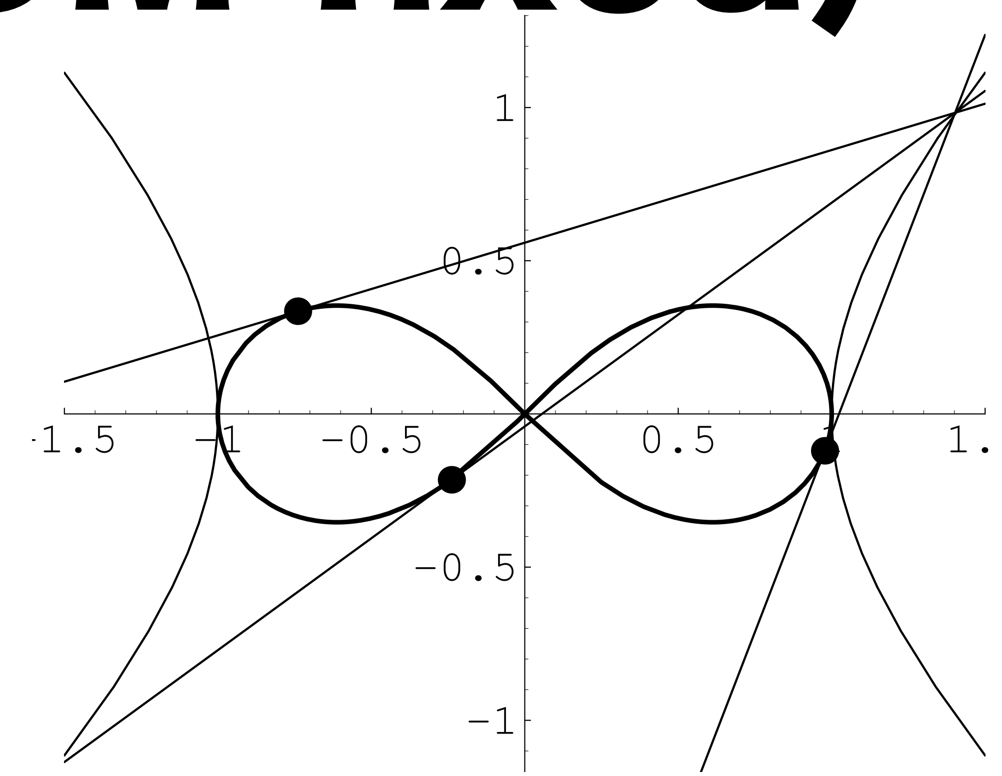
Orbit	Newton	Einstein
	○	× Periastron Shift
	○	○

Fujiwara, Fukuda, Ozaki (2003)

Coplanar 3-body Problem

If total P = 0 (COM fixed)

total L = 0



Tangent lines from 3 bodies always meet at a point

GR figure-8 satisfies 3-tangent line theorem

Because...

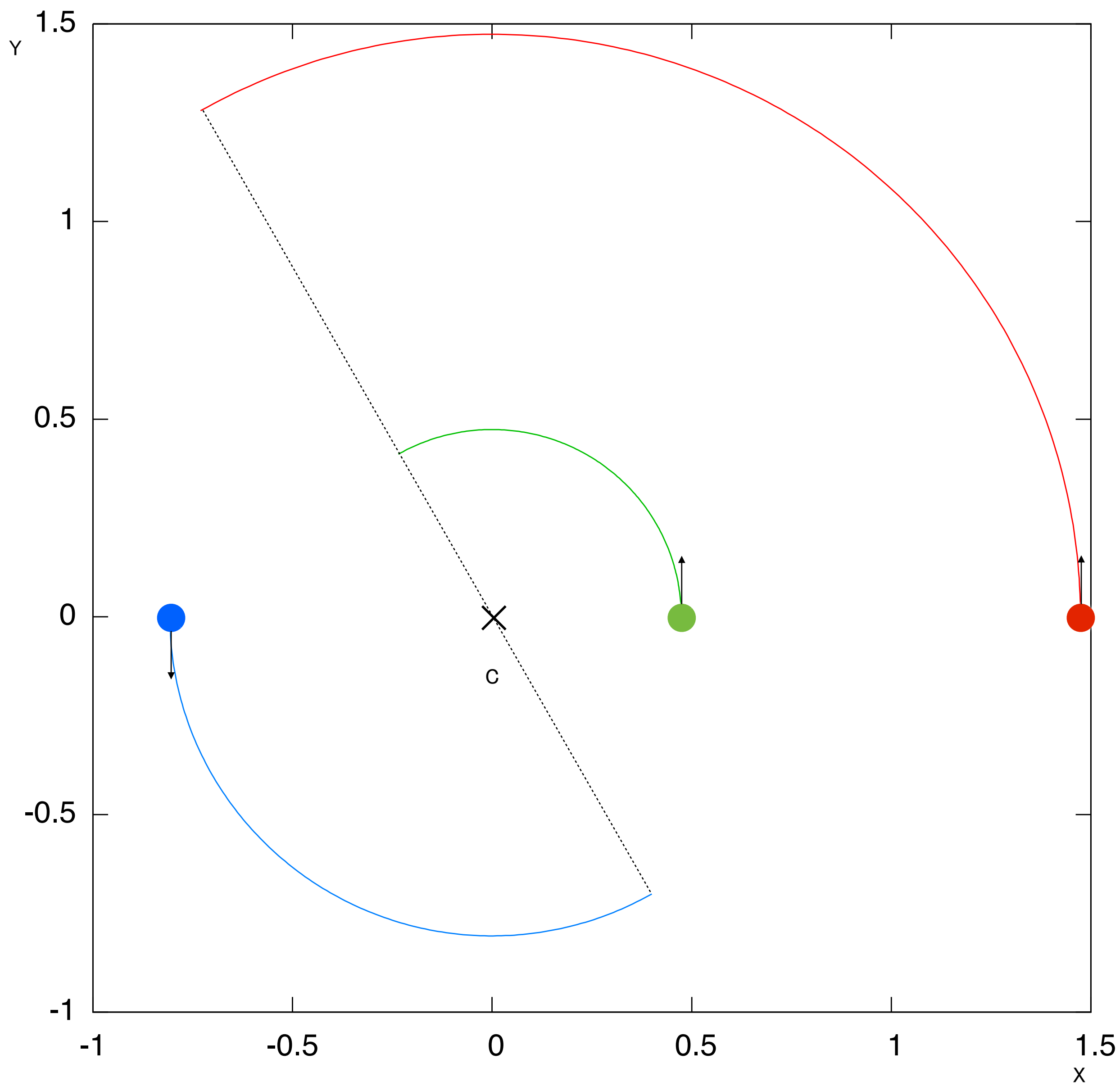
In GR, p and v are not always parallel

In GR figure-8, p and v are parallel

Part 1: Choreography

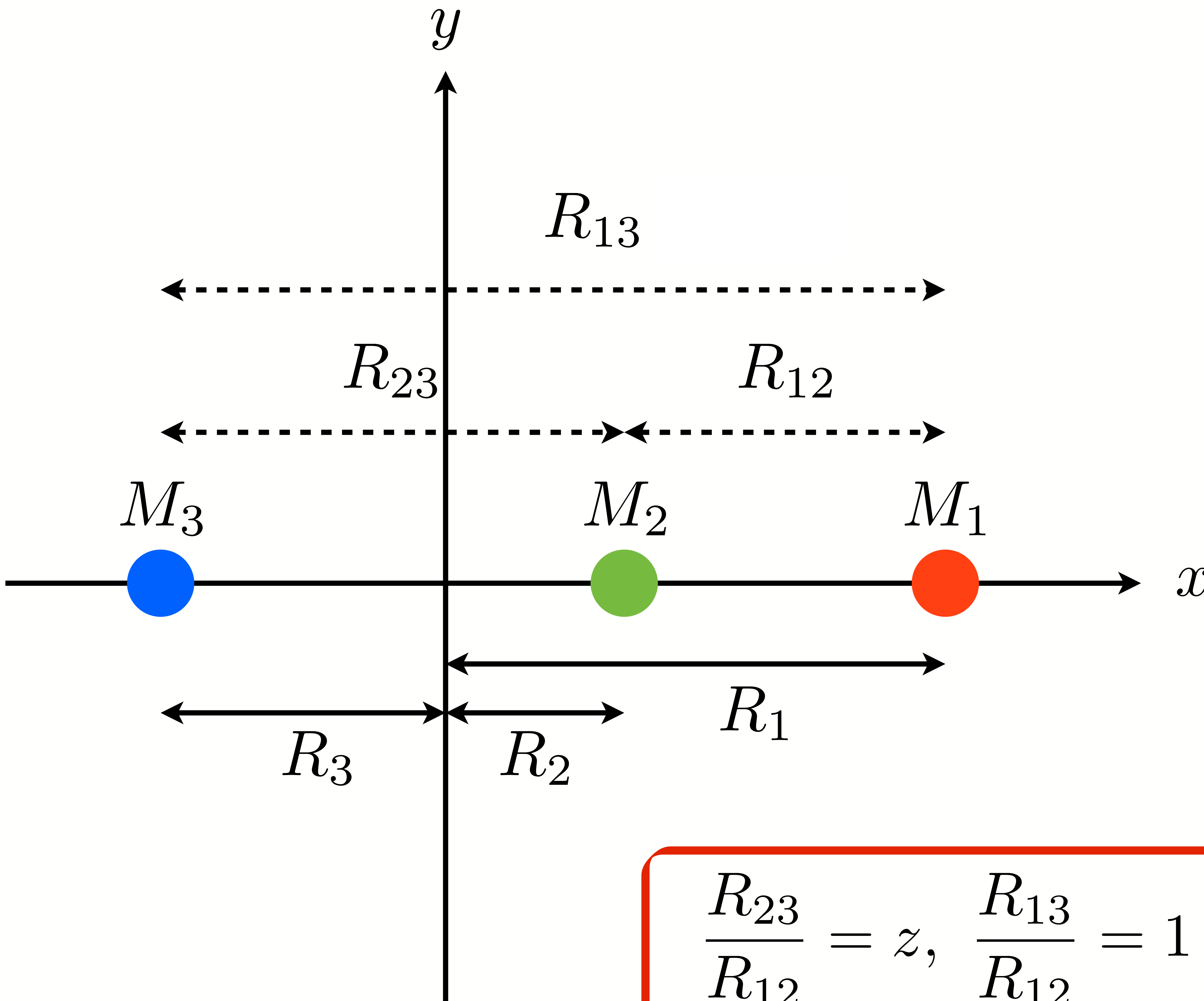
Part 2: Euler+Lagrange's solutions

GR collinear solution



Euler

Three masses always line up



$$\frac{R_{23}}{R_{12}} = z, \quad \frac{R_{13}}{R_{12}} = 1 + z$$

Nonlinear gravity

$$\frac{d^2\boldsymbol{r}_K}{dt^2} = \sum_{A\neq K} \boldsymbol{r}_{AK}\frac{Gm_A}{r_{AK}^3}\left[1 - 4\sum_{B\neq K}\frac{Gm_B}{c^2 r_{BK}} - \sum_{C\neq A}\frac{Gm_C}{c^2 r_{CA}}\left(1 - \frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)\right.$$

$$\left. + \left(\frac{\boldsymbol{v}_K}{c}\right)^2 + 2\left(\frac{\boldsymbol{v}_A}{c}\right)^2 - 4\left(\frac{\boldsymbol{v}_A}{c}\right)\cdot\left(\frac{\boldsymbol{v}_K}{c}\right) - \frac{3}{2}\left(\frac{\left(\frac{\boldsymbol{v}_A}{c}\right)\cdot\boldsymbol{r}_{AK}}{r_{AK}}\right)^2\right]$$

Correction by velocity

$$- \sum_{A\neq K}\left[\left(\frac{\boldsymbol{v}_A}{c}\right) - \left(\frac{\boldsymbol{v}_K}{c}\right)\right]\frac{Gm_A\boldsymbol{r}_{AK}\cdot\left[3\left(\frac{\boldsymbol{v}_A}{c}\right) - 4\left(\frac{\boldsymbol{v}_K}{c}\right)\right]}{r_{AK}^3}$$

$$+ \frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{Gm_C}{r_{CA}^3}\frac{Gm_A}{c^2 r_{AK}}$$

Triple coupling

M1 × M2 × M3

not exist in Newton

Assume

line up

circular motion

Is EIH-EOM satisfied?

Yamada, HA (2010)

$$F(z) \equiv \sum_{k=0}^{7} A_k z^k = 0$$

7th order

$$A_7 = \frac{M}{a}\left[-4 - 2(\nu_1 - 4\nu_3) + 2(\nu_1^2 + 2\nu_1\nu_3 - 2\nu_3^2) - 2\nu_1\nu_3(\nu_1 + \nu_3)\right],$$

$$A_6 = 1 - \nu_3 + \frac{M}{a}\left[-13 - (10\nu_1 - 17\nu_3) + 2(2\nu_1^2 + 8\nu_1\nu_3 - \nu_3^2) + 2(\nu_1^3 - 2\nu_1^2\nu_3 - 3\nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_5 = 2 + \nu_1 - 2\nu_3 + \frac{M}{a}\left[-15 - (18\nu_1 - 5\nu_3) + 4(5\nu_1\nu_3 + 4\nu_3^2) + 6(\nu_1^3 - \nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_4 = 1 + 2\nu_1 - \nu_3 + \frac{M}{a}\left[-6 - 2(5\nu_1 + 2\nu_3) - 4(2\nu_1^2 - \nu_1\nu_3 - 4\nu_3^2) + 2(3\nu_1^3 + \nu_1^2\nu_3 - 2\nu_1\nu_3^2 - 3\nu_3^3)\right],$$

$$A_3 = -(1 - \nu_1 + 2\nu_3) + \frac{M}{a}\left[6 + 2(2\nu_1 + 5\nu_3) - 4(4\nu_1^2 + \nu_1\nu_3 - 2\nu_3^2) + 2(3\nu_1^3 + 2\nu_1^2\nu_3 - \nu_1\nu_3^2 - 3\nu_3^3)\right],$$

$$A_2 = -(2 - 2\nu_1 + \nu_3) + \frac{M}{a}\left[15 - (5\nu_1 - 18\nu_3) - 4(4\nu_1^2 + 5\nu_1\nu_3) + 6(\nu_1^3 + \nu_1^2\nu_3 - \nu_3^3)\right],$$

$$A_1 = -(1 - \nu_1) + \frac{M}{a}\left[13 - (17\nu_1 - 10\nu_3) + 2(\nu_1^2 - 8\nu_1\nu_3 - 2\nu_3^2) + 2(\nu_1^3 + 3\nu_1^2\nu_3 + 2\nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_0 = \frac{M}{a}\left[4 - 2(4\nu_1 - \nu_3) + 2(2\nu_1^2 - 2\nu_1\nu_3 - \nu_3^2) + 2\nu_1\nu_3(\nu_1 + \nu_3)\right].$$

5th order in Newton Gravity

Yamada, HA (2011)

Descartes rule of signs

and

Slow Motion (PN)

$$3 - 2 = 1$$

Uniqueness

(z = positive)

For the same mass and full length, one can show

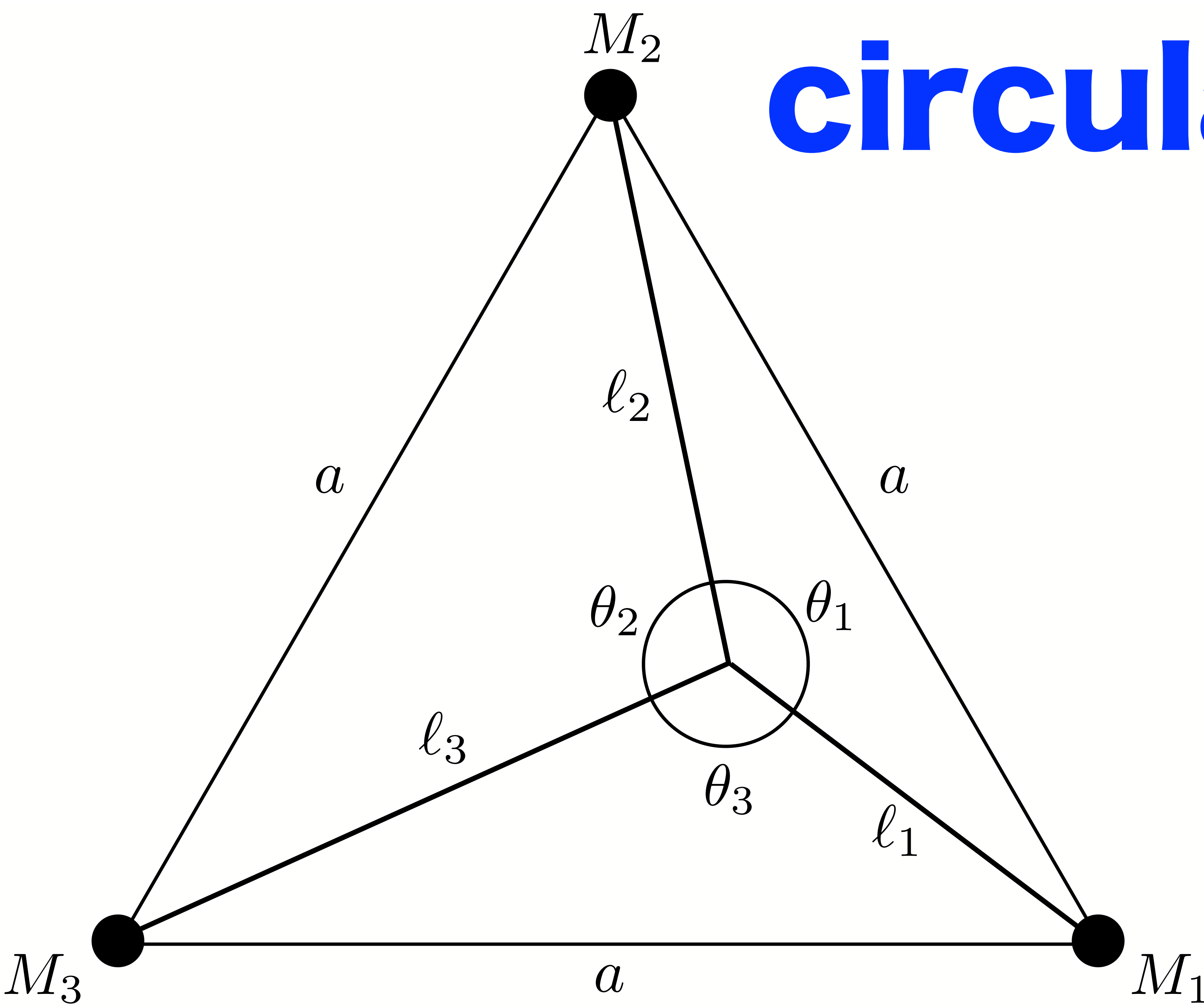
$$\omega < \omega_N$$

GR angular velocity is always smaller

Assume・・

equilateral triangle

circular motion



Equilateral triangular sol.

is possible

in Newton gravity

for three general masses

Ichita, Yamada, HA (2011)

Equilateral triangular sol.

is possible at 1PN in GR

if and only if either

1) Equal finite masses

- mass ratio 1 : 1 : 1

2) One finite,

two test masses

- mass ratio 0 : 0 : 1

A little more...

EOM of M1 becomes

$$
\begin{aligned}
-\omega^2 \boldsymbol{x}_1 \;=\; & -\frac{M}{a^3}\boldsymbol{x}_1 + g_{PN1}\boldsymbol{x}_1 \\
& + \frac{\sqrt{3}M}{16a^3}\boldsymbol{n}_{\perp 1}\frac{M_2 M_3 (M_2 - M_3)}{M_2^2 + M_2 M_3 + M_3^2} \\
& \times \left[10 + \frac{a^3}{M^2}\big(-4M_1 + 5M_2 + 5M_3 \big)\omega^2 \right]
\end{aligned}
$$

M2=M3,

unless test mass

For the same mass and side length, one can show

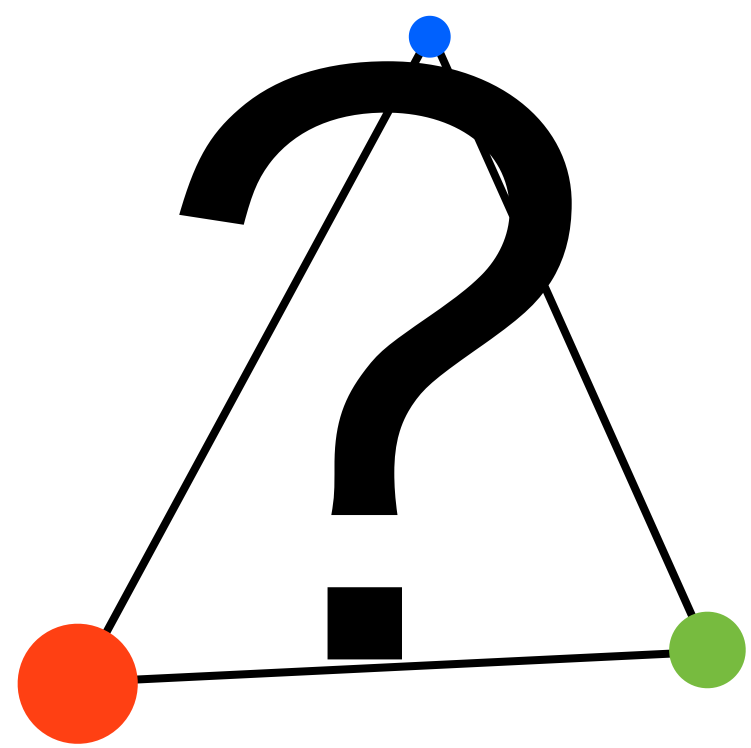
$$\omega < \omega_N$$

GR angular velocity is always smaller

For the arbitrary mass ratio,

a triangular equilibrium solution exist

or not?



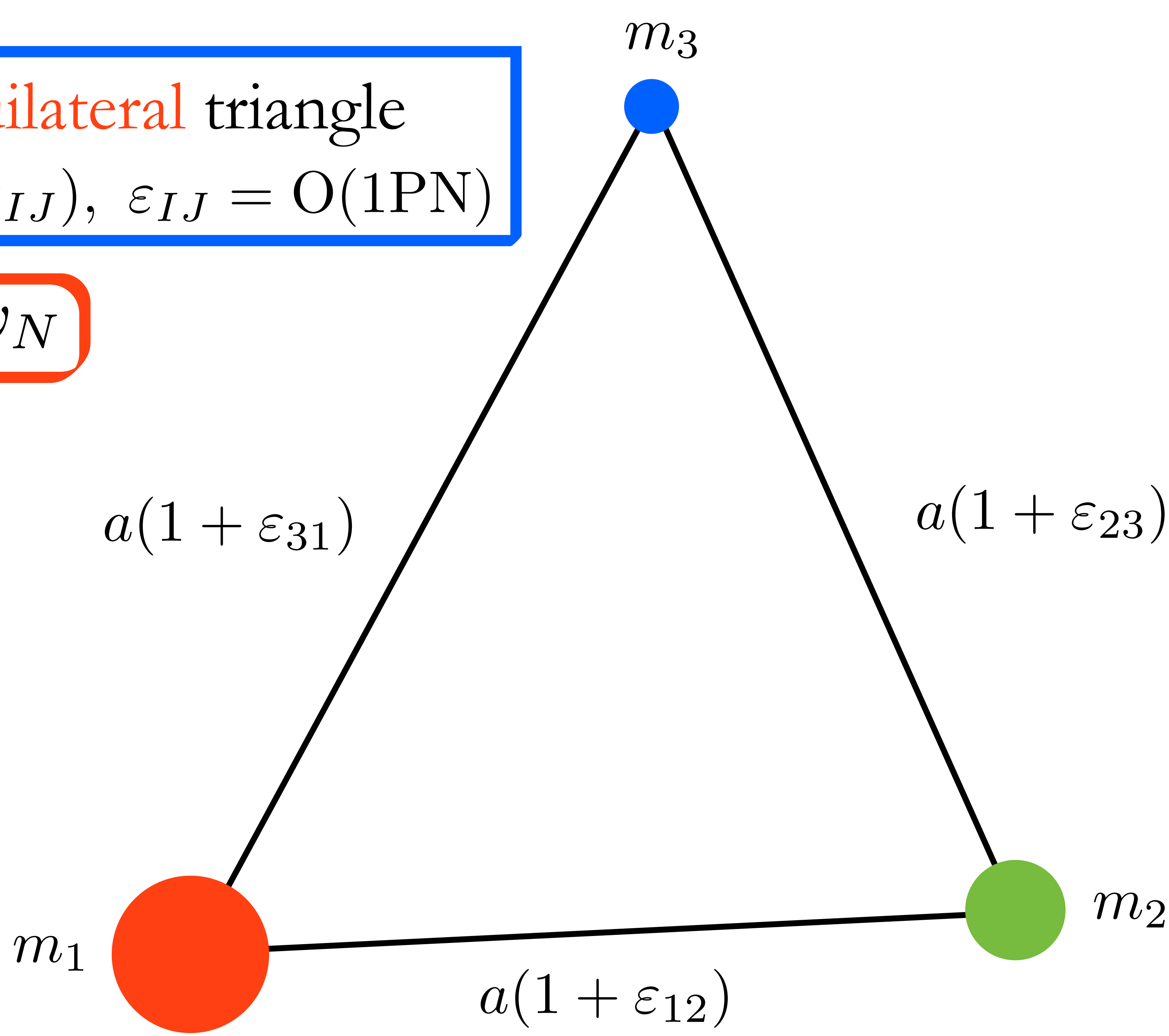
cf. [Krefetz, Astron. J. **72**, 471 (1967)]
for the restricted 3-body problem,
that has been used by [Seto & Muto, PRD **81**, 103004 (2010)]

Corrections of distance

PN inequilateral triangle

$r_{IJ} = a(1+\varepsilon_{IJ}), \ \varepsilon_{IJ} = \mathrm{O}(1\mathrm{PN})$

$\omega = \omega_N$



We can ignore the 1PN correction to the center of mass

Triangular solution at the 1PN

EOM for m_1 becomes

$$
\begin{aligned}
-\omega^2 \boldsymbol{r}_1 = & -\omega_N^2 \boldsymbol{r}_1 \\
& + \nu_2 \left(-3 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 - \frac{3}{8}\nu_3 [5 - 3(\nu_1 + \nu_2)] \right) \lambda \boldsymbol{r}_{21} \\
& + \nu_3 \left(-3 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 - \frac{3}{8}\nu_2 [5 - 3(\nu_3 + \nu_1)] \right) \lambda \boldsymbol{r}_{31} \\
& - 3(\nu_2 \varepsilon_{12} \boldsymbol{r}_{21} + \nu_3 \varepsilon_{31} \boldsymbol{r}_{31})
\end{aligned}
$$

$\omega = \omega_N$ ⇒ [ ] $= 0$

13年12月7日土曜日

As a result,

$$\varepsilon_{12} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_3[5 - 3(\nu_1 + \nu_2)]\right]\lambda,$$

$$\varepsilon_{23} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_1[5 - 3(\nu_2 + \nu_3)]\right]\lambda,$$

$$\varepsilon_{31} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_2[5 - 3(\nu_1 + \nu_3)]\right]\lambda.$$

Triangular solution for the arbitrary mass ratio at 1PN

[Yamada & HA, (2012)]

Solar system

Corrections for L4 (L5) of Solar system [m]

Planet	Sun-Planet	Sun-L4 (L5)	Planet-L4 (L5)
Earth	-1477	-1477	-1477 -923
Jupiter	-1477	-1477	-1477 -922

The sign + corresponds to the increase of distance

§3　Summary

1. Choreography in GR
2. GR extension
   of Euler + Lagrange

There remain a lot of interesting things to do!

Thank you !

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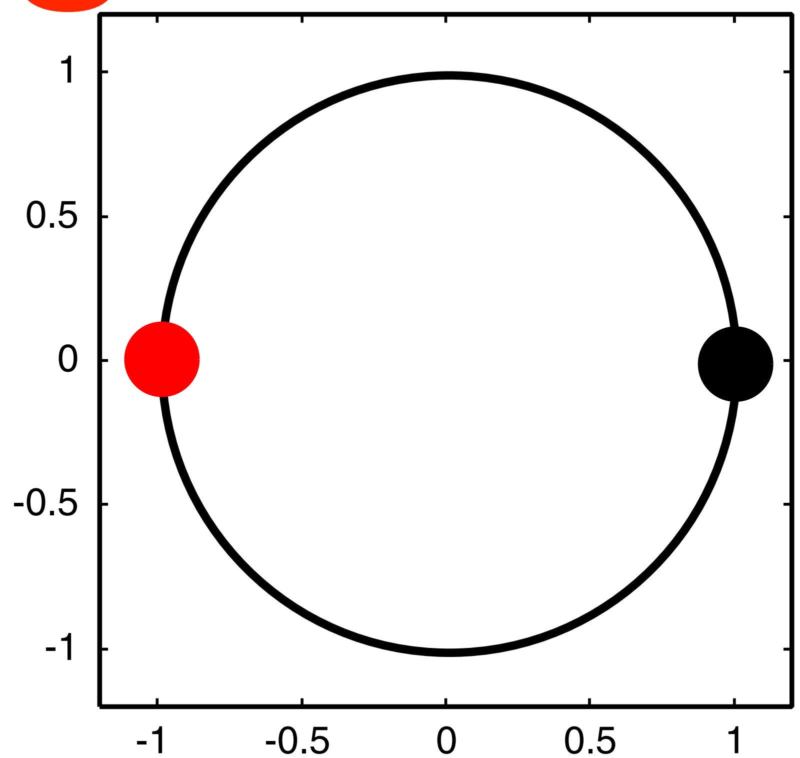
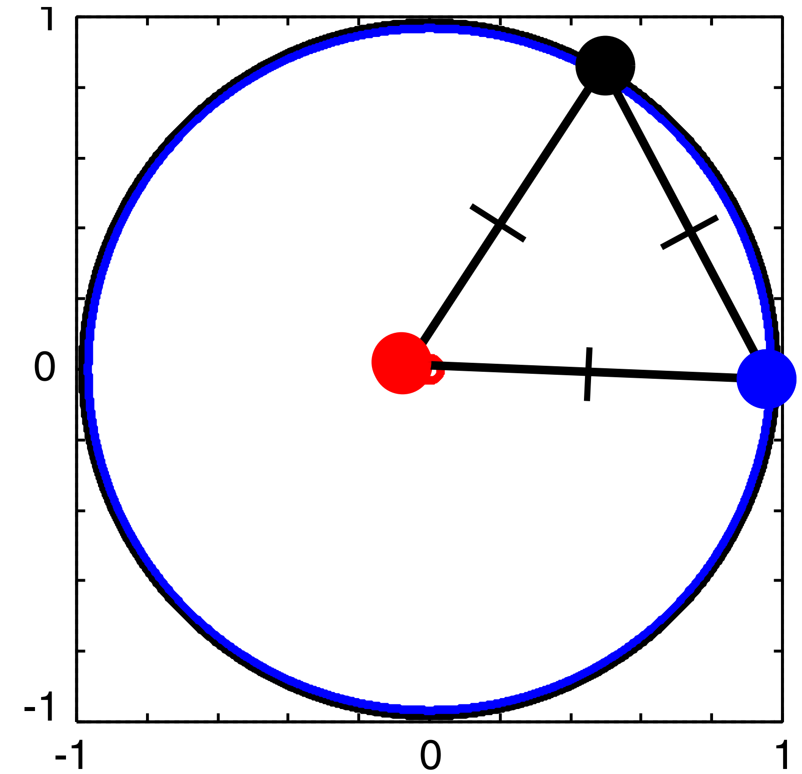
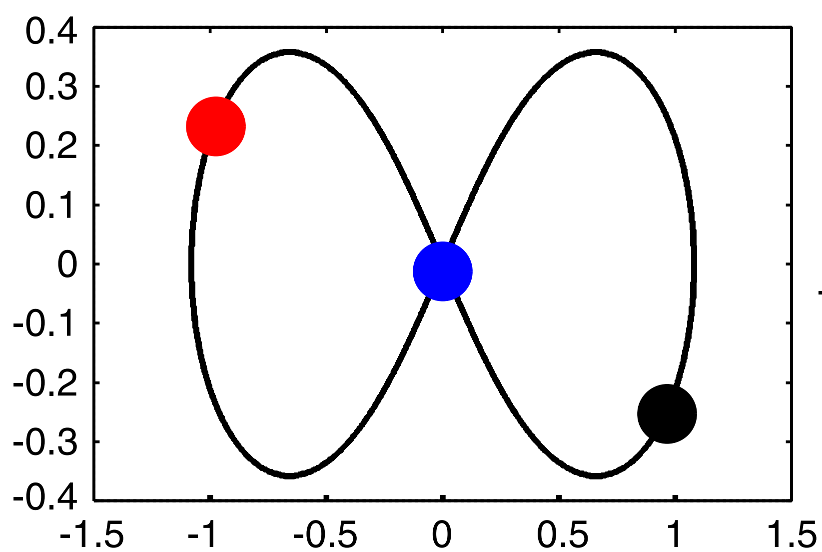
GWs

Torigoe et al. PRL (2009)

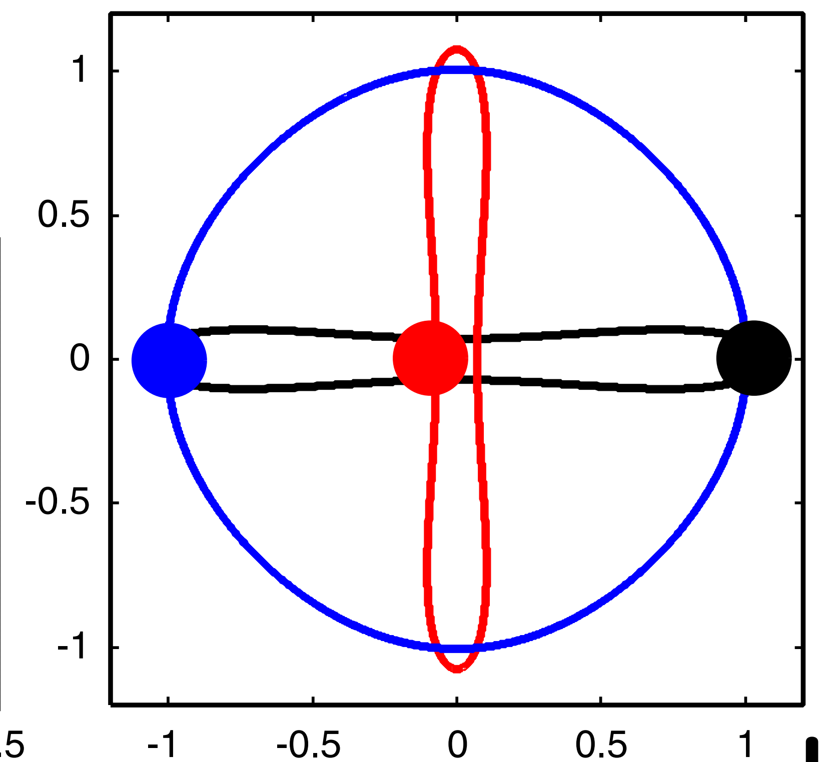




Lagrange's Triangle

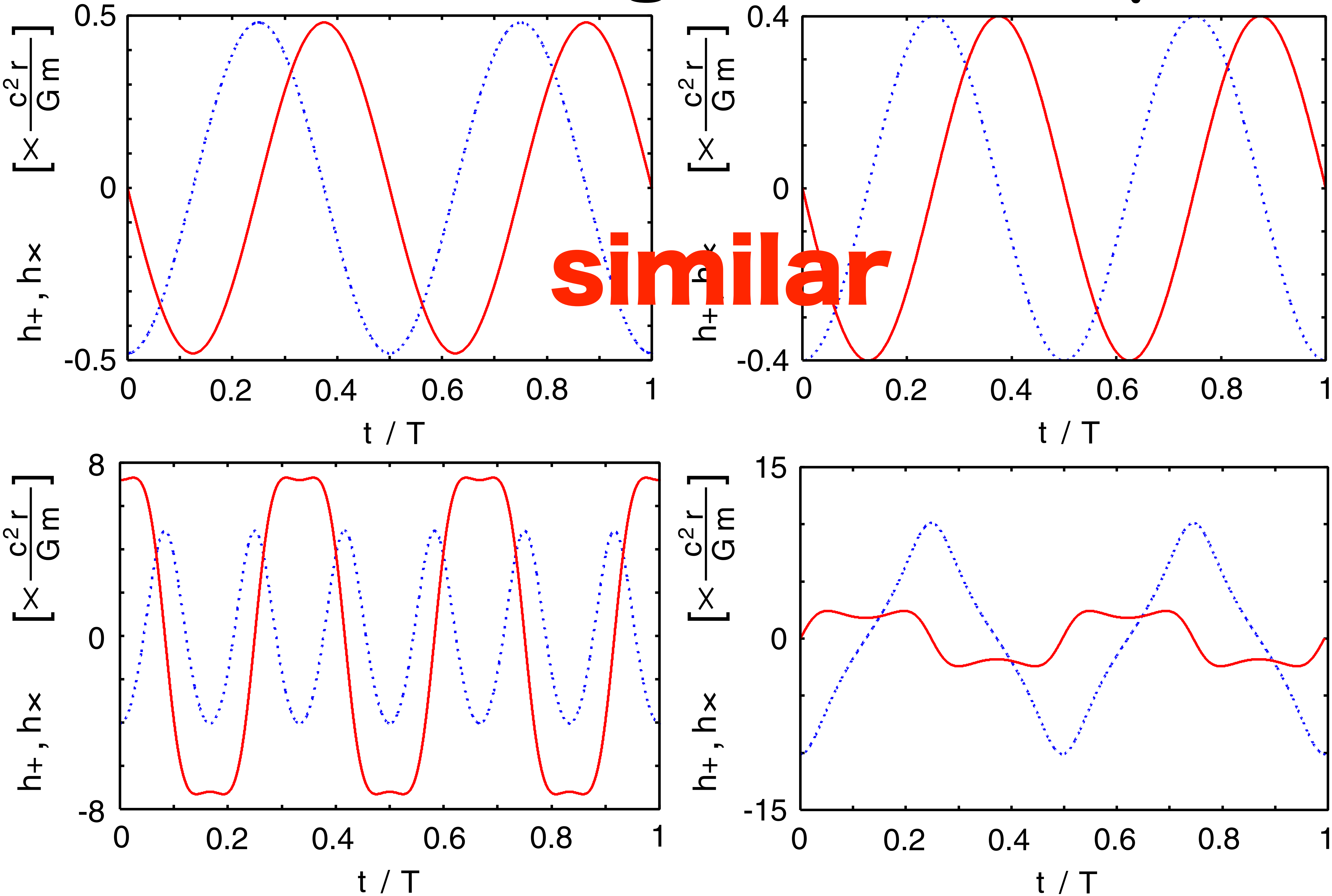


Moore's Figure-8



Henon's Criss-cross

Torigoe et al. PRL (2009)



Obs. z-direction

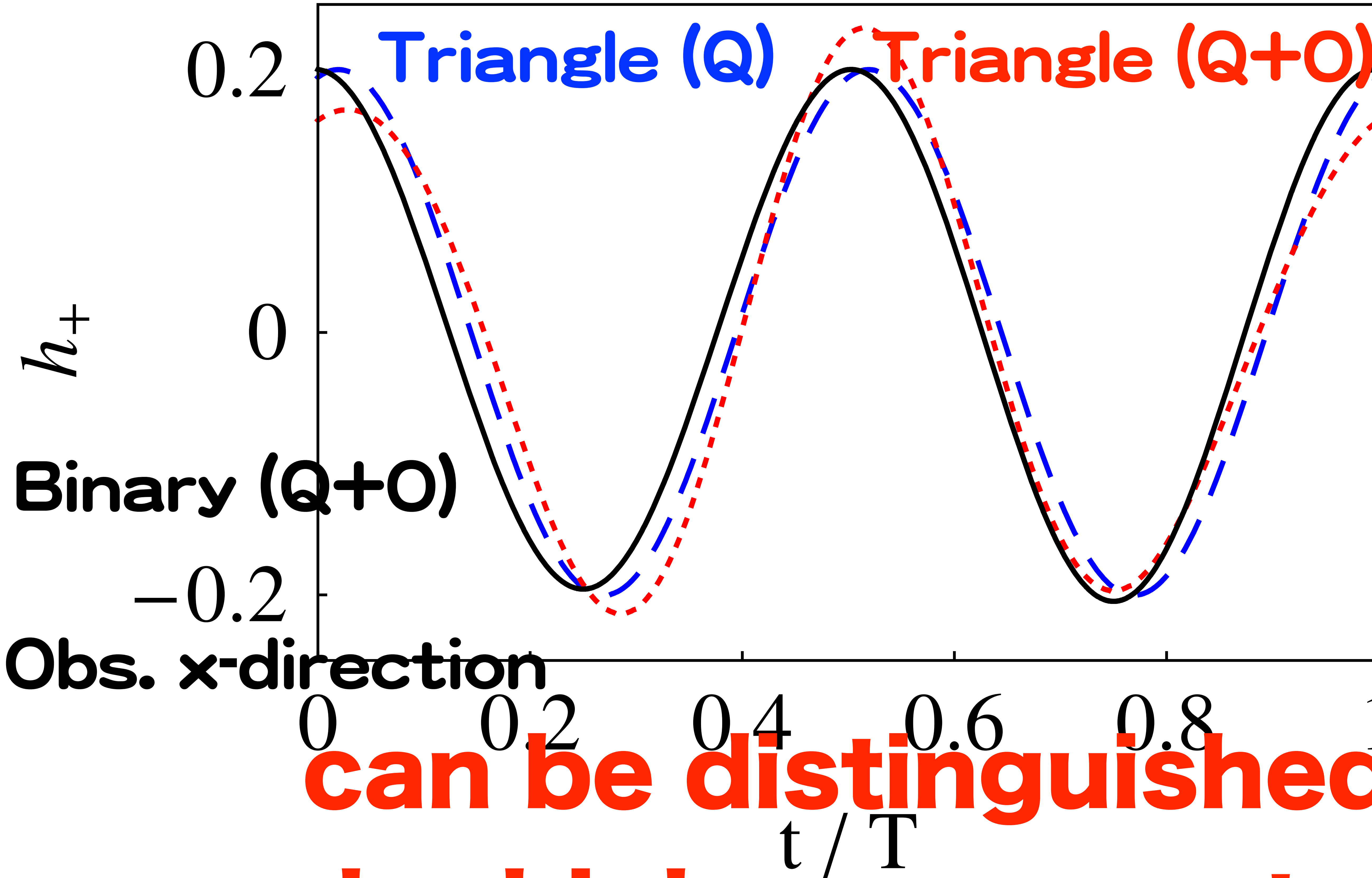
orbital shrinking rate

$$\frac{1}{a}\frac{da}{dt} = -\frac{64}{5}\frac{m_{\rm tot}^3}{a^4}\frac{\left\{\sum_p \nu_p \left(\frac{M_p}{m_{\rm tot}}\right)^{2/3}\right\}^2 - 2\sum_{p\neq q}\nu_p\nu_q\left(\frac{M_p}{m_{\rm tot}}\right)^{2/3}\left(\frac{M_q}{m_{\rm tot}}\right)^{2/3}\sin^2(\theta_p-\theta_q)}{\sum_{p\neq q}\nu_p\nu_q - \sum_p \nu_p\left(\frac{M_p}{m_{\rm tot}}\right)^{2/3}}$$

$$f_{\rm GW}^2 = m_{\rm tot}/\pi^2 a^3$$

$$\frac{1}{f_{\rm GW}}\frac{df_{\rm GW}}{dt} = \frac{96}{5}\pi^{8/3} M_{\rm chirp}^{5/3} f_{\rm GW}^{8/3}$$

same as binary !



can be distinguished by higher moments

Torigoe et al. (2009), HA (2009)

Flow chart

Is GW source a binary?

Paramater determinations of particular 3-body

HA (2009)