Triangular solution to the general relativistic three-body problem

Kei Yamada (D1)
Hirosaki University

with T. Ichita & H. Asada

Contents

Contents

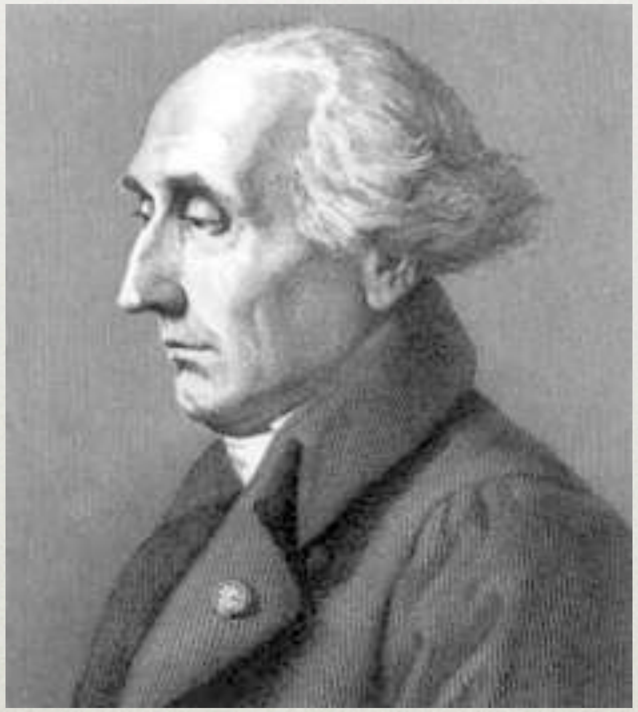
- The background
- Equilateral triangular solution in GR
- Triangular solution in GR: general masses
- Summary

Three-body problem

Particular solutions to the three-body problem

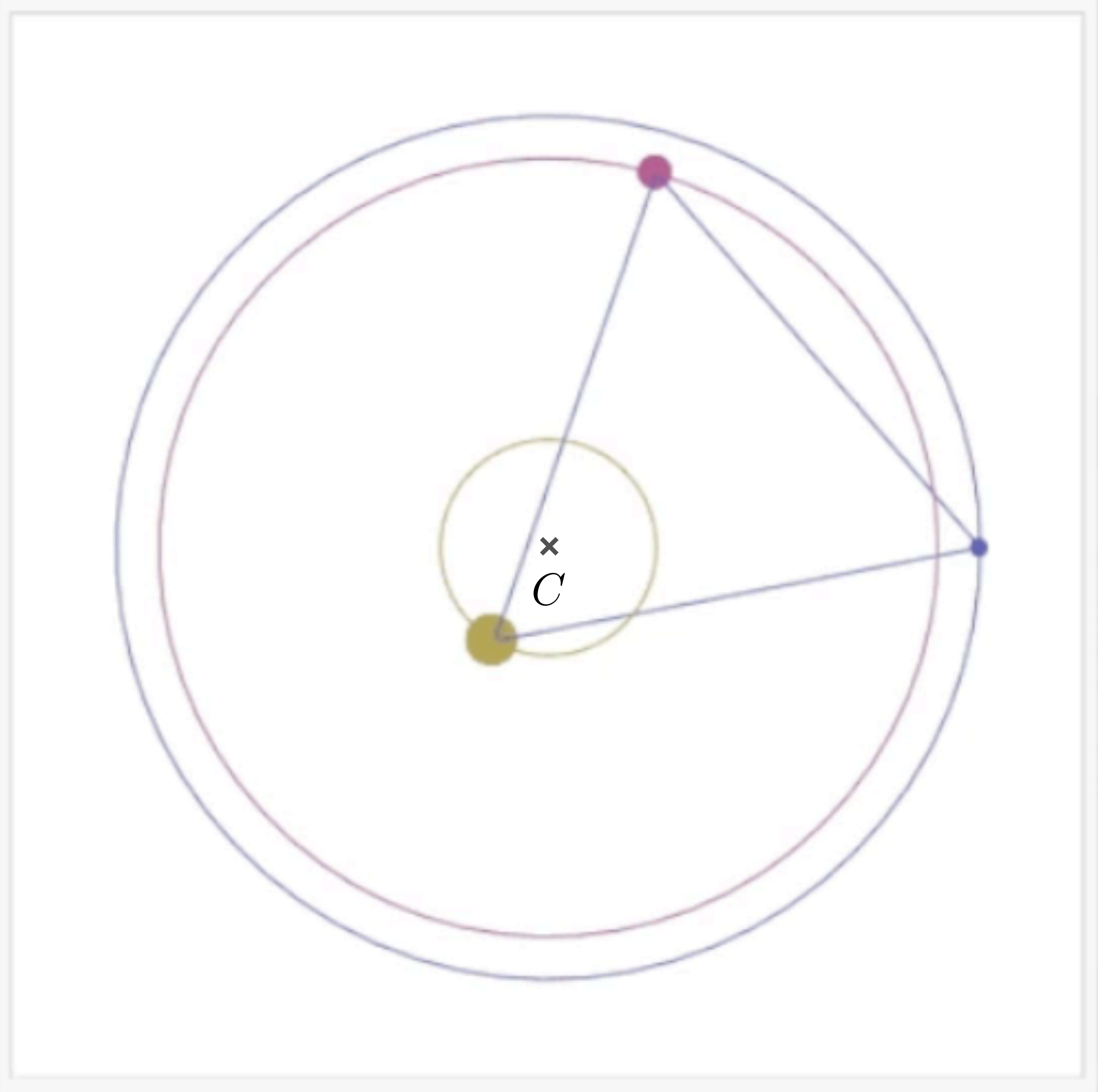
Euler's collinear solution (1765)

&



Lagrange's equilateral triangular solution (1772)

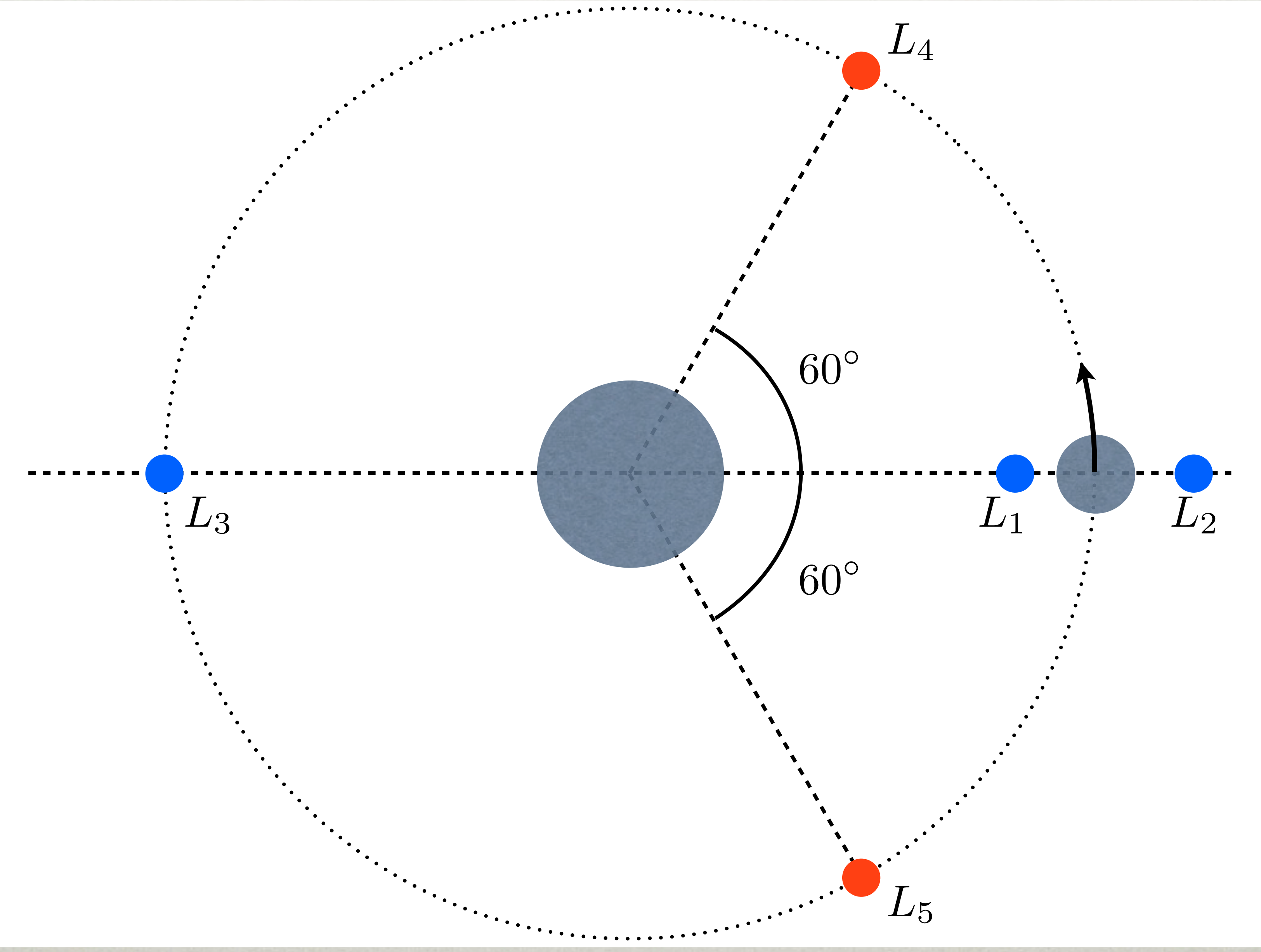
Equilateral triangular solution



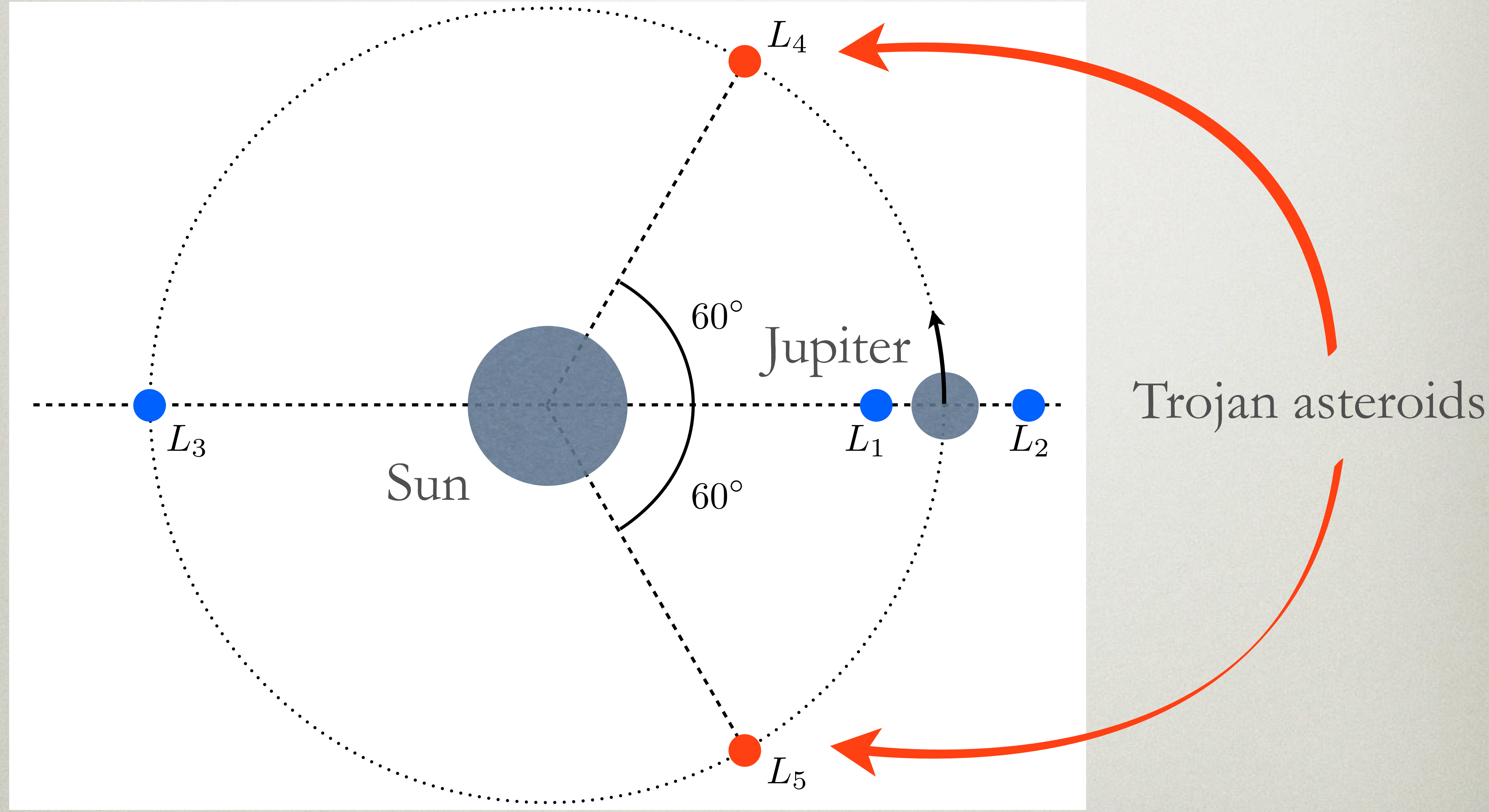
C : the center of mass

Lagrange points

L4 & L5



Lagrange points

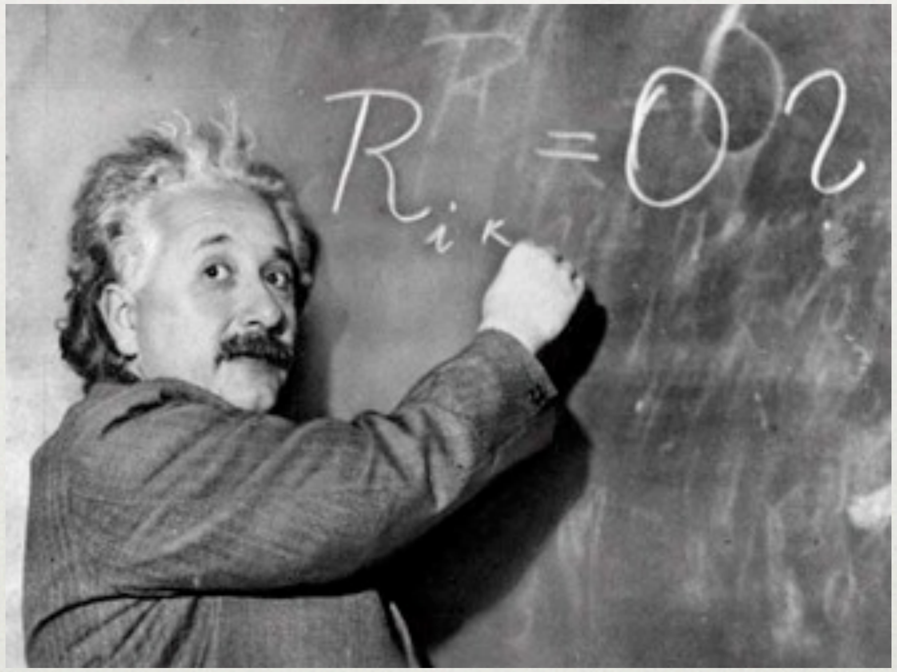


What happens in the general relativity (GR)?

GR effects of Solar system

Two-body system: ○

(e.g. the perihelion precession of Mercury)



Three-body system: ?

It is interesting as a new test of GR

Gravity in GR

Einstein equation

$$G_{\mu\nu} = T_{\mu\nu}$$

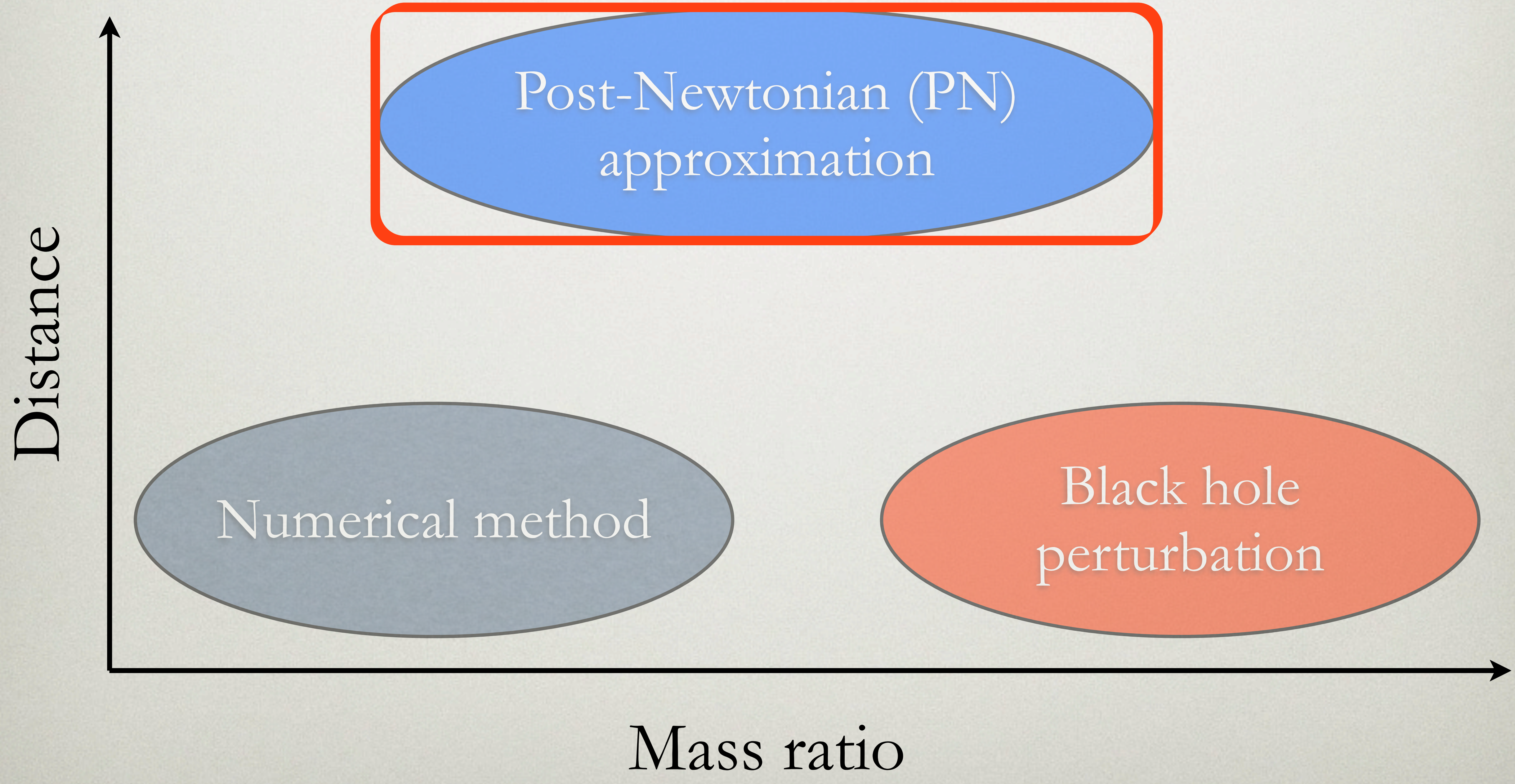
Curvature of Spacetime

Matter

Gravity

10 non-linear second-order partial differential equations

Post-Newtonian approximation



Post-Newtonian approximation

$$\lambda = \left(\frac{v}{c}\right)^2 \sim \frac{GM}{rc^2} \ll 1$$

v : velocity, c : light speed, r : distance, M : mass

Expand Einstein equation

Full order of GR

$$Q = Q_0 + Q_1\lambda + Q_2\lambda^2 + \cdots$$

corrections of GR

Newtonian term

First order of λ = 1PN approximation

EIH equation of motion

Einstein-Infeld-Hoffman (EIH) equation of motion for N bodies

Newtonian term

GR correction by mass

$$
\begin{aligned}
m_K \frac{d^2 \boldsymbol{r}_K}{dt^2} = \sum_{A \neq K} \boldsymbol{r}_{AK} \frac{G m_A m_K}{r_{AK}^3} \Bigg[1 - 4 \sum_{B \neq K} \frac{G m_B}{c^2 r_{BK}} - \sum_{C \neq A} \frac{G m_C}{c^2 r_{CA}} \left(1 - \frac{\boldsymbol{r}_{AK} \cdot \boldsymbol{r}_{CA}}{2 r_{CA}^2} \right) \\
+ \left(\frac{\boldsymbol{v}_K}{c} \right)^2 + 2 \left(\frac{\boldsymbol{v}_A}{c} \right)^2 - 4 \left(\frac{\boldsymbol{v}_A}{c} \right) \cdot \left(\frac{\boldsymbol{v}_K}{c} \right) - \frac{3}{2} \left(\frac{\left(\frac{\boldsymbol{v}_A}{c}\right) \cdot \boldsymbol{r}_{AK}}{r_{AK}} \right)^2 \Bigg] \\
- \sum_{A \neq K} \left[\left(\frac{\boldsymbol{v}_A}{c}\right) - \left(\frac{\boldsymbol{v}_K}{c}\right) \right] \frac{G m_A m_K}{r_{AK}^3} \boldsymbol{r}_{AK} \cdot \left[3 \left(\frac{\boldsymbol{v}_A}{c} \right) - 4 \left(\frac{\boldsymbol{v}_K}{c} \right) \right] \\
+ \frac{7}{2} \sum_{A \neq K} \sum_{C \neq A} \boldsymbol{r}_{CA} \frac{G m_C m_K}{r_{CA}^3} \frac{G m_A}{c^2 r_{AK}}
\end{aligned}
$$

GR correction by velocity

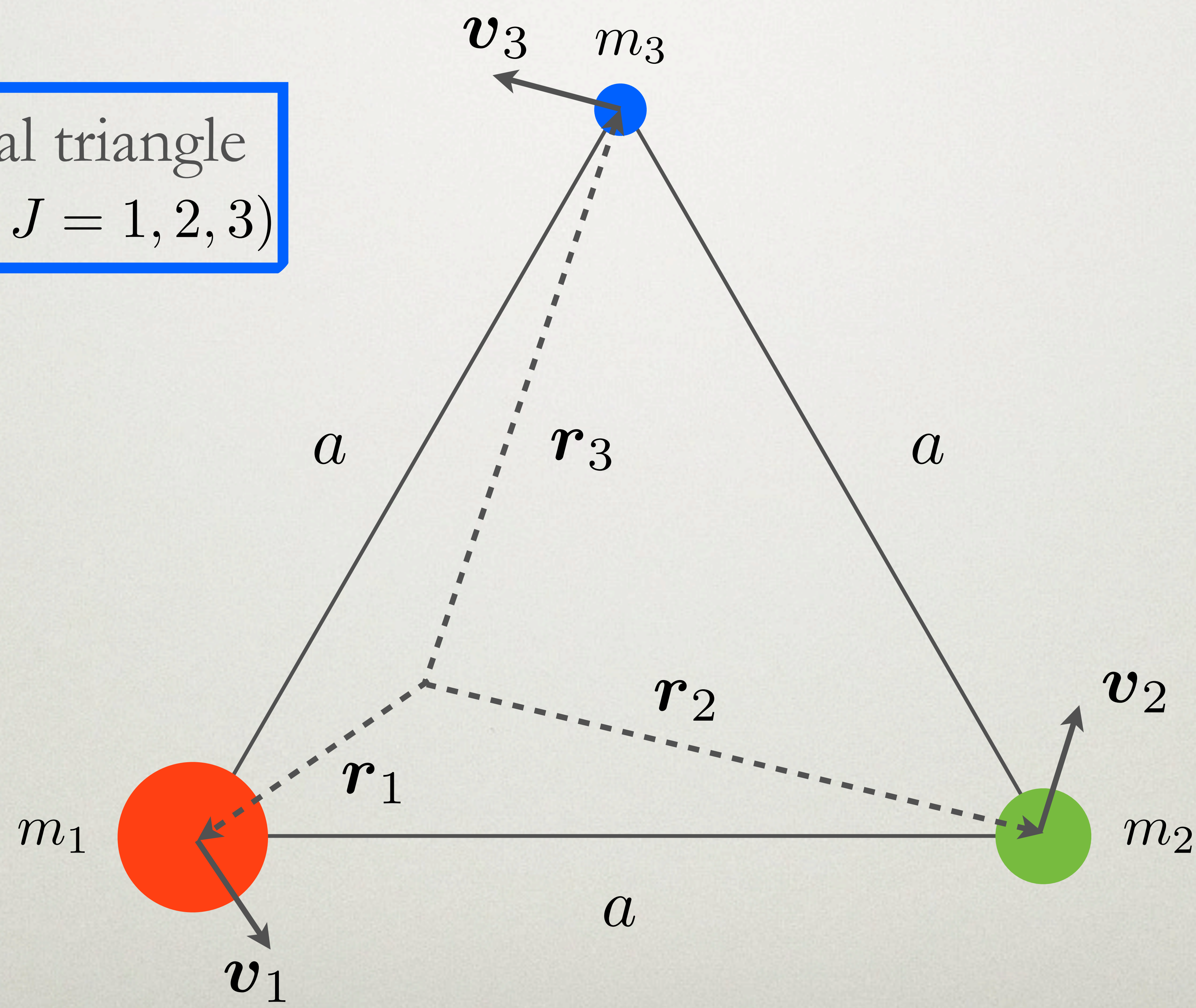
Triple product

We look for an equilibrium solution in a circular motion

Contents

Equilateral triangular configuration

Equilateral triangle
$r_{IJ} = a \ (I, J = 1, 2, 3)$



Center of mass at 1PN

$$\boldsymbol{r}_G = \frac{\sum_A \nu_A \boldsymbol{r}_A \left[1 + \frac{1}{2}\left(\left(\frac{v_A}{c}\right)^2 - \sum_{B\neq A} \frac{Gm_B}{c^2 r_{AB}}\right)\lambda\right]}{\sum_C \nu_C \left[1 + \frac{1}{2}\left(\left(\frac{v_C}{c}\right)^2 - \sum_{D\neq C} \frac{Gm_D}{c^2 r_{CD}}\right)\lambda\right]}$$

In general, it is different from Newtonian one

But

In this case, it agrees with the Newtonian one

Equilateral triangular solution at the 1PN

At 1PN order, EOM for m_1 becomes

$$-\omega^2 \boldsymbol{n}_1 = -\frac{M}{a^3}\boldsymbol{n}_1 + g_{PN1}\boldsymbol{n}_1 + \frac{\sqrt{3}}{16}\frac{M}{a^3}\frac{\nu_2\nu_3(\nu_2-\nu_3)}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\left[6+9\big(\nu_2+\nu_3\big)\right]\lambda\boldsymbol{n}_{\perp 1}$$

$$g_{PN1} = \frac{\lambda}{16(\nu_2^2+\nu_2\nu_3+\nu_3^2)}\frac{M}{a^3} \times \Big[48(\nu_2^2+\nu_2\nu_3+\nu_3^2) - 2(8\nu_2^3+7\nu_2^2\nu_3+7\nu_2\nu_3^2+8\nu_3^3) + (16\nu_2^4+41\nu_2^3\nu_3+84\nu_2^2\nu_3^2+41\nu_2\nu_3^3+16\nu_3^4)\Big]$$

ω : angular velocity, $\nu_I \equiv m_I/M,\ M = \sum_I m_I\ (I=1,2,3)$

$\boldsymbol{n}_1 \equiv \boldsymbol{r}_1/|\boldsymbol{r}_1|,\ \boldsymbol{n}_{\perp 1} \equiv \boldsymbol{v}_1/|\boldsymbol{v}_1|,\ \boldsymbol{n}_{\perp 1}$ is normal to $\boldsymbol{n}_1$

Equilateral triangular solution at the 1PN

$$-\omega^2 \boldsymbol{n}_1 = -\frac{M}{a^3}\boldsymbol{n}_1 + g_{PN1}\boldsymbol{n}_1 + \frac{\sqrt{3}}{16}\frac{M}{a^3}\frac{\nu_2\nu_3(\nu_2-\nu_3)}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\left[6+9(\nu_2+\nu_3)\right]\lambda\boldsymbol{n}_{\perp 1}$$

In only 2 cases, bodies satisfy EOM;

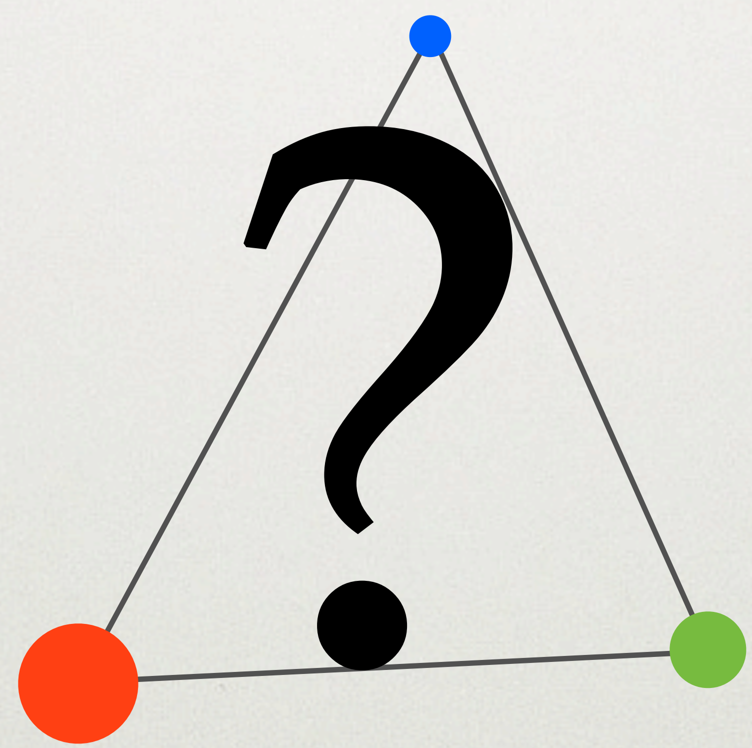
- mass ratio 1 : 1 : 1
- mass ratio 0 : 0 : 1

This solution does not always exist in GR

[Ichita, KY & Asada, PRD **83**, 084026 (2011)]

Equilateral triangular solution at the 1PN

For the arbitrary mass ratio,
the solution exist?



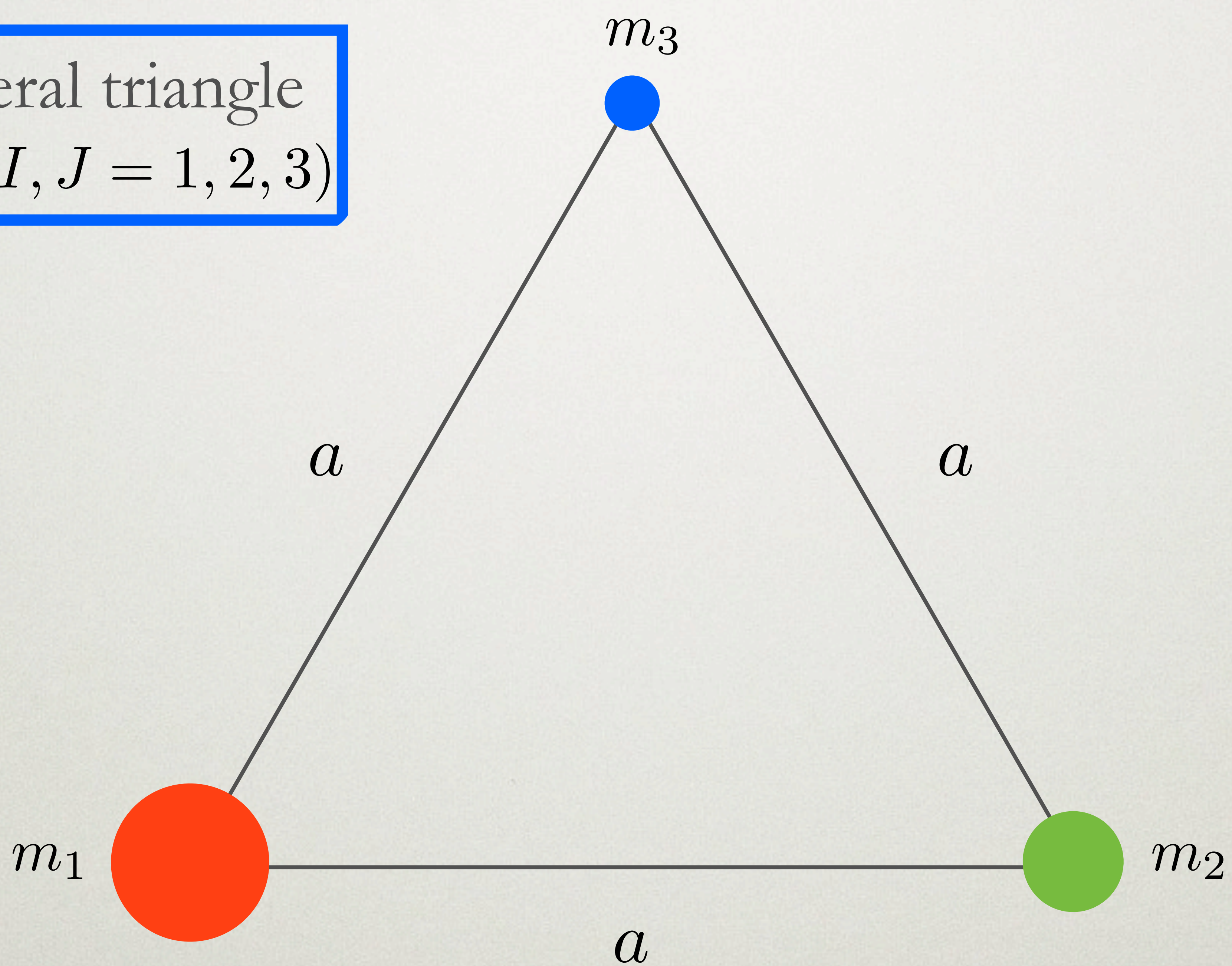
cf. [E. Krefetz, Astron. J. **72**, 471 (1967)]
for restricted 3-body problem

Contents

- The background
- Equilateral triangular solution in GR
- Triangular solution in GR: general masses
- Summary

Corrections of distance

Equilateral triangle
$r_{IJ} = a \; (I, J = 1, 2, 3)$

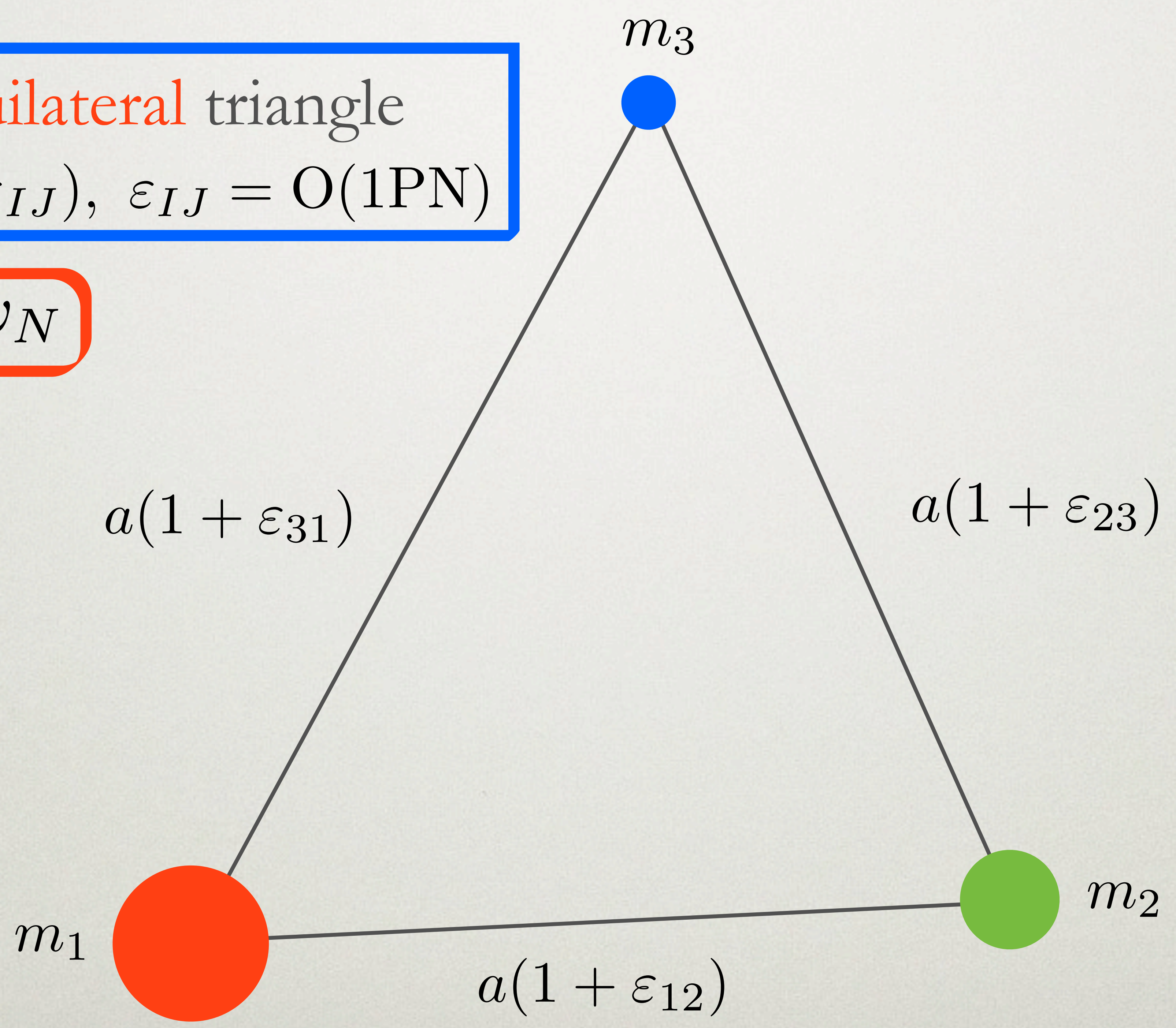


Corrections of distance

PN inequilateral triangle

$r_{IJ} = a(1+\varepsilon_{IJ}), \ \varepsilon_{IJ} = \mathrm{O(1PN)}$

$\omega = \omega_N$



We can ignore the correction for center of mass

Triangular solution at the 1PN

EOM for m_1 becomes

$$
\begin{aligned}
-\omega^2 \boldsymbol{r}_1 = &-\omega_N^2 \boldsymbol{r}_1 \\
&+ \nu_2 \left(-3 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 - \frac{3}{8}\nu_3[5 - 3(\nu_1 + \nu_2)] \right) \lambda \boldsymbol{r}_{21} \\
&+ \nu_3 \left(-3 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 - \frac{3}{8}\nu_2[5 - 3(\nu_3 + \nu_1)] \right) \lambda \boldsymbol{r}_{31} \\
&- 3(\nu_2 \varepsilon_{12} \boldsymbol{r}_{21} + \nu_3 \varepsilon_{31} \boldsymbol{r}_{31})
\end{aligned}
$$

$\omega = \omega_N$ ⇒ $\boxed{} = 0$

Triangular solution at the 1PN

As a result, we could uniquely express ε_{IJ}

$$\varepsilon_{12} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_3[5 - 3(\nu_1 + \nu_2)]\right]\lambda,$$

$$\varepsilon_{23} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_1[5 - 3(\nu_2 + \nu_3)]\right]\lambda,$$

$$\varepsilon_{31} = -\left[1 - \frac{1}{3}(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1) + \frac{1}{8}\nu_2[5 - 3(\nu_1 + \nu_3)]\right]\lambda.$$

⇒ Triangular solution for the arbitrary mass ratio at 1PN

[KY, H. Asada, in prep.]

Application for Solar system

Corrections for L4 (L5) of Solar system [m]

Planet	Sun-Planet	Sun-L4 (L5)	Planet-L4 (L5)
Earth	-1477	-1477	-1477 -923
Jupiter	-1477	-1477	-1477 -922

The sign + denotes increase of distance

Contents

- The background
- Equilateral triangular solution in GR
- Triangular solution in GR: general masses
- Summary

Summary

- We found a triangular solution with 1PN corrections of distance
- The 1PN triangle is smaller than the Newtonian one (for same masses)
- This result may be applied to also near SMBH and compact binary
- The future observations are needed

Future works

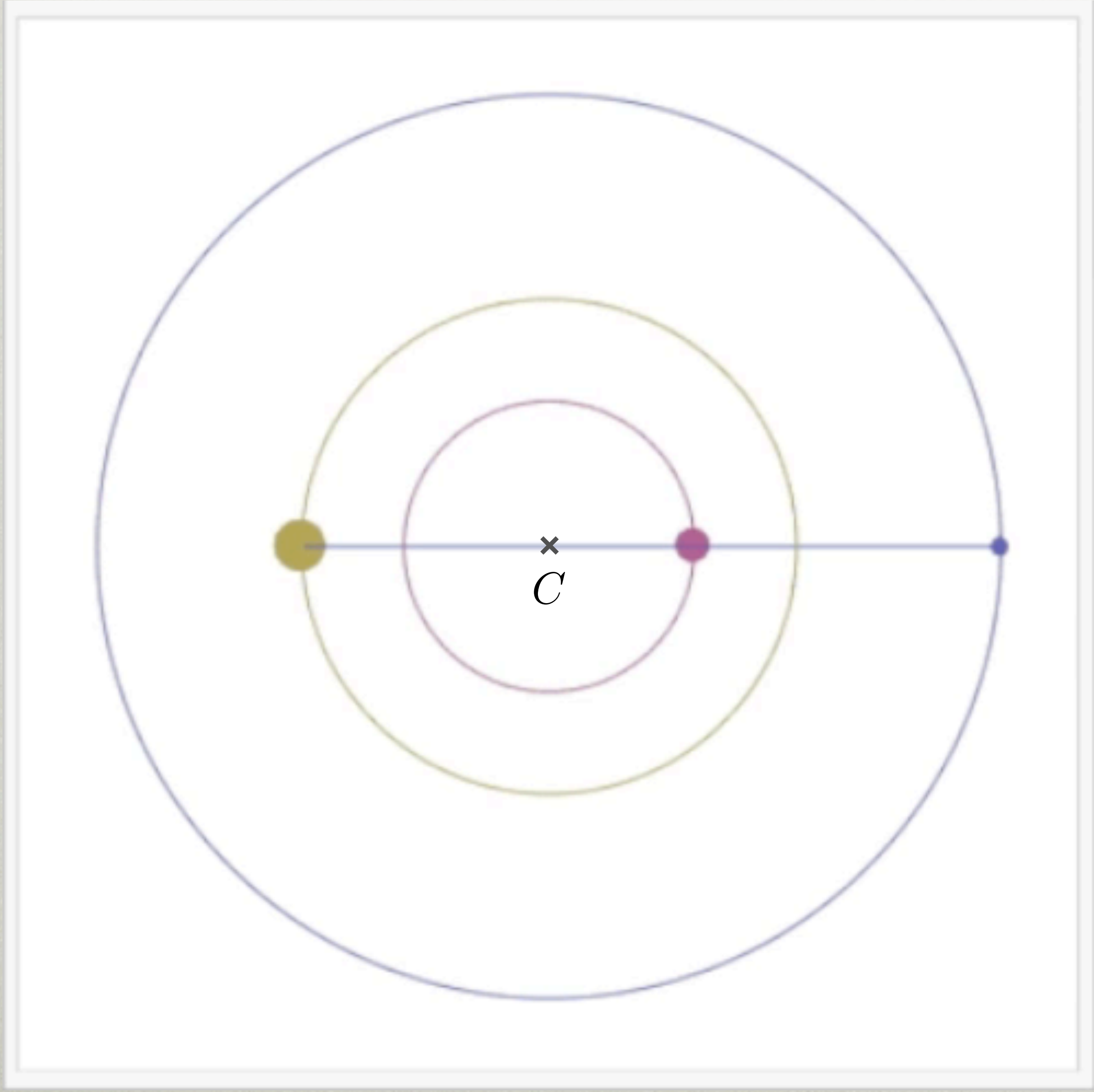
- The Stability
- The Gravitational wave
- An elliptical motion
- More bodies
- Higher order PN approximation

Thank you for your attention

Appendix

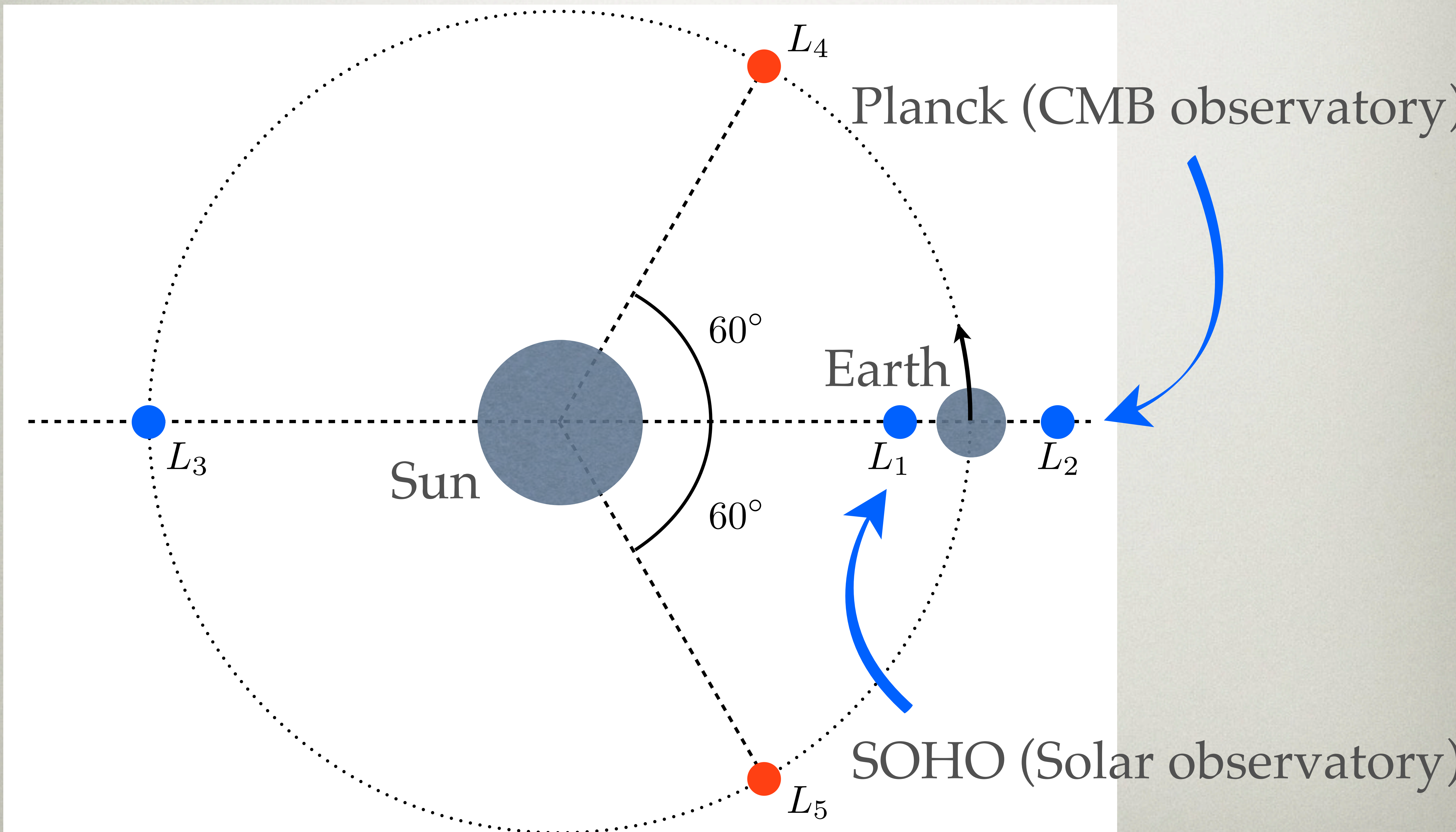
Collinear solution

Collinear solution



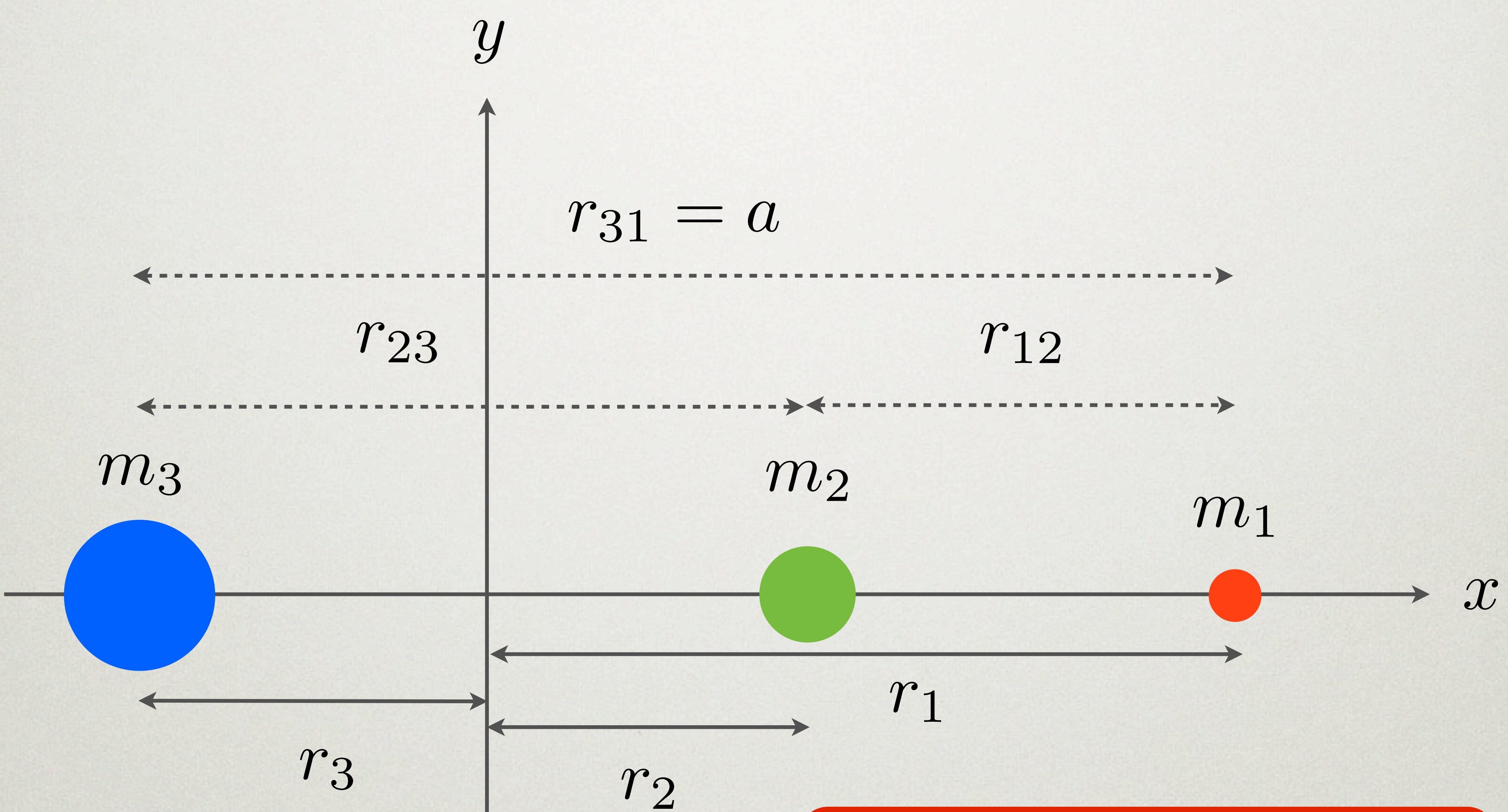
C：共通重心

Lagrange points



Contents

Collinear configuration



$$\frac{r_{23}}{r_{12}} = z, \quad \frac{r_{31}}{r_{12}} = 1 + z$$

Collinear solution in Newtonian

$$\frac{r_{23}}{r_{12}} = z, \quad \frac{r_{31}}{r_{12}} = 1 + z$$

Equations of motion for 3 masses

$$r_1\omega^2 = \left[\frac{m_2}{r_{12}^2} + \frac{m_3}{r_{31}^2}\right], r_2\omega^2 = \left[-\frac{m_1}{r_{12}^2} + \frac{m_3}{r_{23}^2}\right], r_3\omega^2 = \left[-\frac{m_1}{r_{31}^2} - \frac{m_2}{r_{23}^2}\right]$$

$$(m_1 + m_2)z^5 + (3m_1 + 2m_2)z^4 + (3m_1 + m_2)z^3$$
$$-(m_2 + 3m_3)z^2 - (2m_2 + 3m_3)z - (m_2 + m_3) = 0$$

Collinear solution at 1PN

$$F(z) \equiv \sum_{k=0}^{7} A_k z^k = 0$$

$$A_7 = \frac{GM}{ac^2}\left[-4 - 2(\nu_1 - 4\nu_3) + 2(\nu_1^2 + 2\nu_1\nu_3 - 2\nu_3^2) - 2\nu_1\nu_3(\nu_1 + \nu_3)\right],$$

$$A_6 = 1 - \nu_3 + \frac{GM}{ac^2}\left[-13 - (10\nu_1 - 17\nu_3) + 2(2\nu_1^2 + 8\nu_1\nu_3 - \nu_3^2) + 2(\nu_1^3 - 2\nu_1^2\nu_3 - 3\nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_5 = 2 + \nu_1 - 2\nu_3 + \frac{GM}{ac^2}\left[-15 - (18\nu_1 - 5\nu_3) + 4(5\nu_1\nu_3 + 4\nu_3^2) + 6(\nu_1^3 - \nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_4 = 1 + 2\nu_1 - \nu_3 + \frac{GM}{ac^2}\left[-6 - 2(5\nu_1 + 2\nu_3) - 4(2\nu_1^2 - \nu_1\nu_3 - 4\nu_3^2) + 2(3\nu_1^3 + \nu_1^2\nu_3 - 2\nu_1\nu_3^2 - 3\nu_3^3)\right],$$

$$A_3 = -(1 - \nu_1 + 2\nu_3) + \frac{GM}{ac^2}\left[6 + 2(2\nu_1 + 5\nu_3) - 4(4\nu_1^2 + \nu_1\nu_3 - 2\nu_3^2) + 2(3\nu_1^3 + 2\nu_1^2\nu_3 - \nu_1\nu_3^2 - 3\nu_3^3)\right],$$

$$A_2 = -(2 - 2\nu_1 + \nu_3) + \frac{GM}{ac^2}\left[15 - (5\nu_1 - 18\nu_3) - 4(4\nu_1^2 + 5\nu_1\nu_3) + 6(\nu_1^3 + \nu_1^2\nu_3 - \nu_3^3)\right],$$

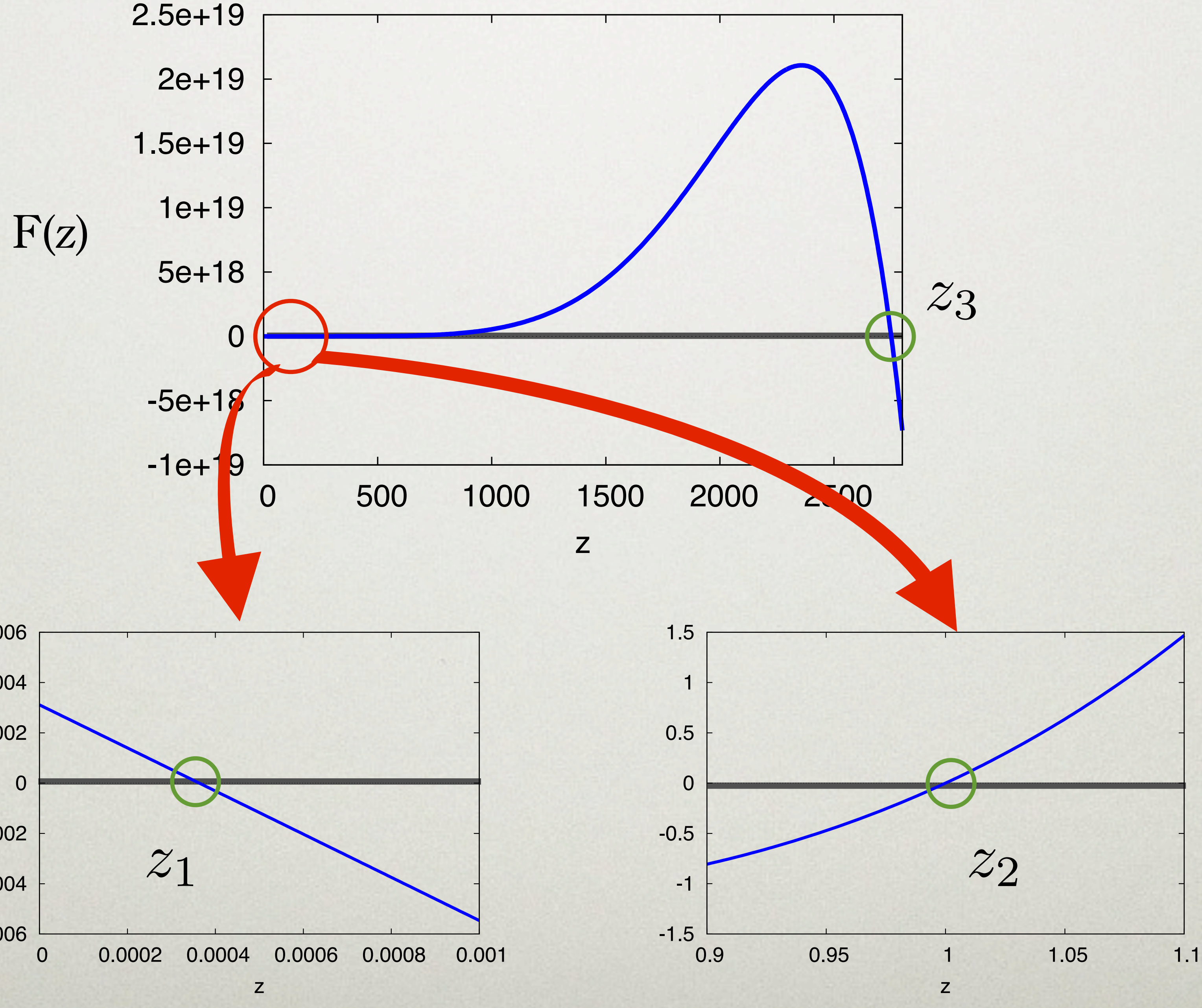
$$A_1 = -(1 - \nu_1) + \frac{GM}{ac^2}\left[13 - (17\nu_1 - 10\nu_3) + 2(\nu_1^2 - 8\nu_1\nu_3 - 2\nu_3^2) + 2(\nu_1^3 + 3\nu_1^2\nu_3 + 2\nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_0 = \frac{GM}{ac^2}\left[4 - 2(4\nu_1 - \nu_3) + 2(2\nu_1^2 - 2\nu_1\nu_3 - \nu_3^2) + 2\nu_1\nu_3(\nu_1 + \nu_3)\right].$$

[Yamada & Asada, PRD 82, 104019 (2010)]

7th-order equation

$$\nu_1 = 1/7, \nu_2 = 5/7, \nu_3 = 1/7, \lambda = GM/c^2 a = 10^4$$



7th-order equation

7th-order equation may have 3 positive roots

Smallest root and largest one give fast motion, which is comparable to speed of light.

# of physical roots = 1

PN angular velocity

PN angular velocity is expressed as

$$\omega = \omega_N \left(1 + \frac{F_M}{2F_N} + \frac{F_V}{2R_{31}}\right),$$

$$F_N = \frac{M}{a^2 z^2}\left[(\nu_1+\nu_3)z^2 + (1-\nu_1-\nu_3)(1+z^2)(1+z)^2\right],$$

$$\begin{aligned}
F_M = -\frac{M^2}{a^3 z^3}\Big[&(4-4\nu_1+\nu_3)(1-\nu_1-\nu_3) \\
&+ (12-7\nu_1+3\nu_3)(1-\nu_1-\nu_3)z \\
&+ (12-\nu_1+\nu_3)(1-\nu_1-\nu_3)z^2 \\
&+ (8-7\nu_1-7\nu_3+8\nu_1\nu_3+3\nu_1^2+3\nu_3^2)z^3 \\
&+ (12+\nu_1-\nu_3)(1-\nu_1-\nu_3)z^4 \\
&+ (12+3\nu_1-7\nu_3)(1-\nu_1-\nu_3)z^5 \\
&+ (4+\nu_1-4\nu_3)(1-\nu_1-\nu_3)z^6\Big],
\end{aligned}$$

$$\begin{aligned}
F_V = \frac{M}{(1+z)^2 z^2}\Big[&-\nu_1^2(1-\nu_1-\nu_3) \\
&- 2\nu_1(1+\nu_1-\nu_3)(1-\nu_1-\nu_3)z \\
&+ (2-2\nu_1+\nu_3+6\nu_1\nu_3-3\nu_3^2+\nu_1^3-3\nu_1^2\nu_3-3\nu_1\nu_3^2+\nu_3^3)z^2 \\
&+ 2(2-\nu_1-\nu_3)(1+\nu_1+\nu_3-\nu_1^2+\nu_1\nu_3-\nu_3^2)z^3 \\
&+ (2+\nu_1-2\nu_3-3\nu_1^2+6\nu_1\nu_3+\nu_1^3-3\nu_1^2\nu_3-3\nu_1\nu_3^2+\nu_3^3)z^4 \\
&- 2\nu_3(1-\nu_1+\nu_3)(1-\nu_1-\nu_3)z^5 \\
&- \nu_3^2(1-\nu_1-\nu_3)z^6\Big].
\end{aligned}$$

ω_N : Newtonian angular velocity

PN angular velocity

For same masses and full length $a (= r_{31})$

we can prove inequality

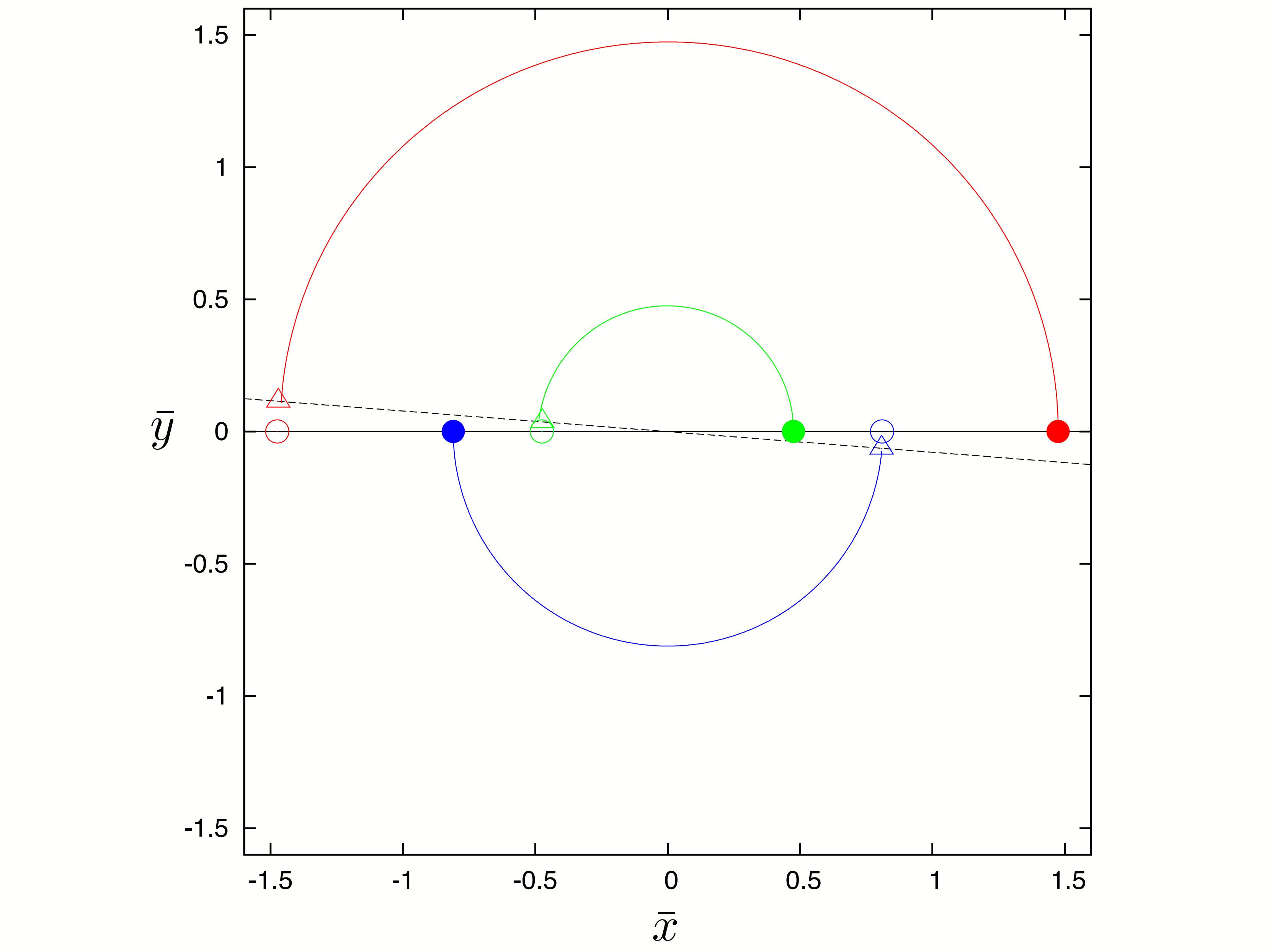
$$\omega < \omega_N$$

Thus, PN angular velocity is always smaller than Newtonian

Namely, PN orbital period is always longer than Newtonian

Numerical example

$$\nu_1 : \nu_2 : \nu_3 = 1 : 2 : 3,\ \lambda = 10^{-2}\ (\frac{v}{c} \sim 10^{-1})$$



[Yamada & Asada, PRD 83, 024040 (2011)]