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Two 2.5D models for 2 vortex problem

Mai Kameda, Misato Nakabayashi, and Masayoshi Sekiguchi[1]

Kisarazu National College of Technology

Abstract. The well known model for N-vortex dynamics treats a vortex as a point on the plane or other 2D surfaces. We propose new two models for point vortex dynamics on the plane in which we take vertical influence into account, that is, 2.5D models. In this paper we just prepare the foundation of these models.

1. Introduction

The so-called N vortex problem is described by the following system:

$$\dot{x}_j = -\sum_{k\neq j}^{N} \frac{\Gamma_k (y_j - y_k)}{r_{jk}^2},\ \dot{y}_j = \sum_{k\neq j}^{N} \frac{\Gamma_k (x_j - x_k)}{r_{jk}^2}, \quad (j = 1, \ldots, N), \quad (1)$$

where (x_j, y_j) is the coordinate of j-th vortex, $\Gamma_j \in \mathbb{R}\backslash\{0\}$ is the *vorticity*, $r_{jk}^2 = (x_j - x_k)^2 + (y_j - y_k)^2$ and $(\dot{\ }) = \dfrac{d}{dt}$. If we introduce a linear transformation:

$$z_j = x_j + iy_j, \qquad w_j = x_j - iy_j,$$

then we have

$$\dot{z}_j = i\sum_{k\neq j}^{N} \frac{\Gamma_k}{w_j - w_k},\ \dot{w}_j = -i\sum_{k\neq j}^{N} \frac{\Gamma_k}{z_j - z_k}, \quad (j = 1, \ldots, N), \quad (2)$$

with $i = \sqrt{-1}$. This linear transformation is regular, hence the systems (1) and (2) are equivalent to each other. Equation (2) can be written

[1]masa@kisarazu.ac.jp

down in the Hamiltonian form through the Legendre transformation of the Lagarangian (see [Newton, 2000, p. 22]):

$$L = \sum_{j=1}^{N} \Gamma_j \left[\frac{\dot{z}_j w_j - z_j \dot{w}_j}{2i} - \sum_{k \neq j} \Gamma_k \ln(z_j - z_k)(w_j - w_k) \right]. \tag{3}$$

The conjugate momenta with respect to z_j and w_j are

$$p_j \equiv \frac{\partial L}{\partial \dot{z}_j} = \frac{\Gamma_j}{2i} w_j, \qquad \pi_j \equiv \frac{\partial L}{\partial \dot{w}_j} = -\frac{\Gamma_j}{2i} z_j,$$

respectively. Then, we have the corresponding Hamiltonian:

$$H(z, w) = \sum_{j=1}^{N} \sum_{k \neq j} \Gamma_j \Gamma_k \ln(z_j - z_k)(w_j - w_k), \tag{4}$$

where $z = (z_1, \cdots, z_N)$ and $w = (w_1, \cdots, w_N)$. Note that

$$H(z, w) = H\left(-\frac{2i\pi}{\Gamma}, \frac{2ip}{\Gamma}\right) = \widetilde{H}(p, \pi),$$

and its canonical equations are

$$\frac{dz_j}{dt} = \frac{\partial \widetilde{H}}{\partial p_j}, \ \frac{dw_j}{dt} = \frac{\partial \widetilde{H}}{\partial \pi_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial z_j}, \ \frac{d\pi_j}{dt} = -\frac{\partial H}{\partial w_j}.$$

Actually, it is sufficient to use the latter two of these equations in order to study N vortex problem.

Angular momentum M is a constant along the motion. In fact,

$$\begin{aligned} M &= \sum_{j=1}^{N} \Gamma_j (x_j \dot{y}_j - \dot{x}_j y_j) \\ &= \frac{1}{2i} \sum_{j=1}^{N} \Gamma_j (\dot{z}_j w_j - z_j \dot{w}_j) \\ &= \sum_{j>k} \Gamma_j \Gamma_k, \end{aligned}$$

where we use Equation (2). Note that M appears in the Lagrangian L as the first term. Similarly, moment of inertia

$$I = \sum_{j=1}^{N} \Gamma_j (x_j^2 + y_j^2) = \sum_{j=1}^{N} \Gamma_j z_j w_j$$

is constant along the motion because

$$\dot{I} = \sum_{j=1}^{N} \Gamma_j (\dot{z}_j w_j + z_j \dot{w}_j) = 0,$$

by using Equation (2).

2. Layered Model

We suppose that vortex filament intersects several layers where the intersections appear as point vortex. Vertical distance between two vorteces on the nearest layers is constant. That is, vortex is treated as piece-wise linear filament with several point vorteces on independent layers under the restriction on the vertical distance. We call this model the *layered model.*

2.1. Double Filaments with Double Layers

Here the case with two filaments bridged over two layers will be discussed (see Figure 1).

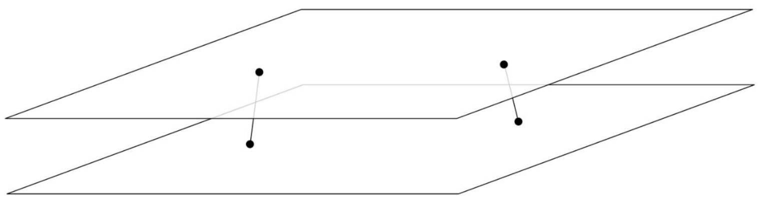

Figure 1. Double Layers Model
Point vorteces indicated by points are connected with sticks.

This model assumes two vorteces on each layer, that is, the first and the second vortex z_1 and z_2 on the first layer, and the other two vorteces z_3 and z_4 on the second layer. Similarly to (10), we have the Lagrangian with constraint as follows:

$$\begin{aligned} \mathcal{L} &= \sum_{j=1}^{4} \frac{\Gamma_j}{2i} (\dot{z}_j w_j - z_j \dot{w}_j) - (\Gamma_1 + \Gamma_2) \ln(z_2 - z_1)(w_2 - w_1) \\ & \quad -(\Gamma_3 + \Gamma_4) \ln(z_4 - z_3)(w_4 - w_3) \\ & \quad +\lambda_1 (D_1^2 - d_1^2) + \lambda_2 (D_2^2 - d_2^2), \end{aligned} \tag{5}$$

where λ_1 and λ_2 are Lagrange's multipliers, and

$$D_1 \equiv |z_3 - z_1|, \quad D_2 \equiv |z_4 - z_2|, \tag{6}$$

and d_1 and d_2 the corresponding values, respectively. Here we introduce the relative coordinate $\zeta \equiv (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$:

$$\zeta_1 \equiv z_2 - z_1, \;\; \zeta_2 \equiv z_4 - z_3, \;\; \zeta_3 \equiv z_3 - z_1, \;\; \zeta_4 \equiv z_4 - z_2. \tag{7}$$

This can be regarded as a linear transformation $z \mapsto \zeta$ with its domain: $\sum_{k=1}^{4} \Gamma_k z_k = 0$. Here we write z as a row vector (z_1, z_2, z_3, z_4). This transformation looks singular. However this is invertible iff $(\Gamma_1+\Gamma_4)(\Gamma_2+\Gamma_3)(\sum_k \Gamma_k) \neq 0$ because the image space: $\zeta_1 - \zeta_2 - \zeta_3 + \zeta_4 = 0$ as well as the domain: $\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 + \Gamma_4 z_4 = 0$ is 3D subspace. In fact, we have the following inverse transformation:

$$z_1 = \frac{(\Gamma_1 - \Gamma)(\Gamma_2\zeta_1 + \Gamma_3\zeta_3) - \Gamma_4(\Gamma_3\zeta_2 + \Gamma_2\zeta_4)}{(\Gamma_2+\Gamma_3)\Gamma}, \tag{8}$$

$$z_2 = \frac{(\Gamma - \Gamma_2)(\Gamma_1\zeta_1 - \Gamma_4\zeta_4) + \Gamma_3(\Gamma_4\zeta_2 - \Gamma_1\zeta_3)}{(\Gamma_1+\Gamma_4)\Gamma}, \tag{9}$$

$$z_3 = \frac{\Gamma_2(\Gamma_4\zeta_4 - \Gamma_1\zeta_1) - (\Gamma - \Gamma_3)(\Gamma_4\zeta_2 - \Gamma_1\zeta_3)}{(\Gamma_1+\Gamma_4)\Gamma}, \tag{10}$$

$$z_4 = \frac{\Gamma_1(\Gamma_2\zeta_1 + \Gamma_3\zeta_3) + (\Gamma - \Gamma_4)(\Gamma_3\zeta_2 + \Gamma_2\zeta_4)}{(\Gamma_2+\Gamma_3)\Gamma}, \tag{11}$$

where $\Gamma = \sum_k \Gamma_k$. Let us introduce a matrix $C = (c_{ij})$ for convenience such that

$$z^t = C\zeta^t, \tag{12}$$

where A^t indicates the transposed matrix of A. Similarly, we introduce the relative coordinate $\varrho \equiv (\varrho_1, \varrho_2, \varrho_3, \varrho_4)$:

$$\varrho_1 \equiv w_2 - w_1, \;\; \varrho_2 \equiv w_4 - w_3, \;\; \varrho_3 \equiv w_3 - w_1, \;\; \varrho_4 \equiv w_4 - w_2. \tag{13}$$

We regard this equation as a linear transformation $w \mapsto \varrho$ with its domain: $\sum_{k=1}^{4} \Gamma_k w_k = 0$. Here we write w as a row vector (w_1, w_2, w_3, w_4). This transformation is also regular iff $(\Gamma_1 + \Gamma_4)(\Gamma_2 + \Gamma_3)\Gamma \neq 0$. The inverse transformation is given by

$$w^t = C\varrho^t. \tag{14}$$

The first term of the Lagrangian $\mathcal{L}$ is the angular momentum $\mathcal{M}$. This is also transformed through the linear transformation (7) and (13) as follows.

$$2i\mathcal{M} = \sum_{j=1}^{4} \Gamma_j(\dot{z}_j w_j - z_j \dot{w}_j) = \dot{z}Gw^t - zG\dot{w}^t = \dot{\zeta}\mathcal{G}\varrho^t - \zeta\mathcal{G}\dot{\varrho}^t$$

with G =diag$(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ and $\mathcal{G} = C^t G C$. The matrix $\mathcal{G}$ becomes a diagonal matrix again:

$$\mathcal{G} = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4), \tag{15}$$

with

$$\begin{aligned} \gamma_1 &= \left(\frac{(\Gamma - \Gamma_1)\Gamma_2}{(\Gamma_2 + \Gamma_3)\Gamma}\right)^2 \Gamma_1, \quad \gamma_2 = \left(\frac{\Gamma_3\Gamma_4}{(\Gamma_1 + \Gamma_4)\Gamma}\right)^2 \Gamma_2, \\ \gamma_3 &= \left(\frac{(\Gamma - \Gamma_3)\Gamma_1}{(\Gamma_1 + \Gamma_4)\Gamma}\right)^2 \Gamma_3, \quad \gamma_4 = \left(\frac{(\Gamma - \Gamma_4)\Gamma_2}{(\Gamma_2 + \Gamma_3)\Gamma}\right)^2 \Gamma_4. \end{aligned} \tag{16}$$

Then we have

$$\mathcal{M} = \frac{1}{2i}\sum_{j=1}^{4} \gamma_j (\dot{\zeta}_j \varrho_j - \zeta_j \dot{\varrho}_j). \tag{17}$$

In addition, we have

$$D_1^2 = |\zeta_3|^2 = \zeta_3 \varrho_3, \quad D_2^2 = |\zeta_4|^2 = \zeta_4 \varrho_4, \tag{18}$$

Thus, the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \mathcal{M} - (\Gamma_1 + \Gamma_2) \ln \zeta_1 \varrho_1 - (\Gamma_3 + \Gamma_4) \ln \zeta_2 \varrho_2 \\ &\quad + \lambda_1 (\zeta_3 \varrho_3 - d_1^2) + \lambda_2 (\zeta_4 \varrho_4 - d_2^2), \end{aligned} \tag{19}$$

with d_1 and d_2 positive constant. Thus the following Euler-Lagrange equations are obtained.

$$\dot{\zeta}_1 = i\frac{\Gamma_1 + \Gamma_2}{\gamma_1 \varrho_1}, \ \dot{\zeta}_2 = i\frac{\Gamma_3 + \Gamma_4}{\gamma_2 \varrho_2}, \ \dot{\zeta}_3 = -i\frac{\lambda_1}{\gamma_3}\zeta_3, \ \dot{\zeta}_4 = -i\frac{\lambda_2}{\gamma_4}\zeta_4, \tag{20}$$

$$\dot{\varrho}_1 = -i\frac{\Gamma_1 + \Gamma_2}{\gamma_1 \zeta_1}, \ \dot{\varrho}_2 = -i\frac{\Gamma_3 + \Gamma_4}{\gamma_2 \zeta_2}, \ \dot{\varrho}_3 = i\frac{\lambda_1}{\gamma_3}\varrho_3, \ \dot{\varrho}_4 = i\frac{\lambda_2}{\gamma_4}\varrho_4. \tag{21}$$

The corresponding Hamiltonian is given as follows:

$$\begin{aligned} \mathcal{H} &= (\Gamma_1 + \Gamma_2) \ln \zeta_1 \varrho_1 + (\Gamma_3 + \Gamma_4) \ln \zeta_2 \varrho_2 \\ &\quad - \lambda_1 (\zeta_3 \varrho_3 - d_1^2) - \lambda_2 (\zeta_4 \varrho_4 - d_2^2), \end{aligned} \tag{22}$$

which can be obtained from (17), (18), (20), and (21). through the Legendre transformation. The Hamiltonian is a constant along motion because it does not explicitly depend on time t. General solution of (20) and (21) can be derived easily. Integral constants appeared in the general solution are choosen so as to satisfy the condition on the total energy, angular momentum, linear momentum and moment of inertia.

We ca obtain the following relation quite algebraically:

$$\begin{aligned}
\sum_{k=1}^{4} \gamma_k \dot{\zeta}_k \varrho_k &= \dot{\zeta}\mathcal{G}\varrho = \dot{z}Gw^t = \sum_{k=1}^{4} \Gamma_k \dot{z}_k w_k \\
&= \sum_{k=1}^{4} \Gamma_k \{x_k \dot{x}_k + y_k \dot{y}_k + i(x_k \dot{y}_k - \dot{x}_k y_k)\} \\
&= \dot{\mathcal{I}} + i\mathcal{M}, \qquad (23)
\end{aligned}$$

where $\mathcal{I}$ is the momentum of inertia. On the other hand, we have

$$\sum_{k=1}^{4} \gamma_k \dot{\zeta}_k \varrho_k = i(\Gamma - \lambda_1 d_1^2 - \lambda_2 d_2^2)$$

by using Equations (20) and (21). This implies $\mathcal{I}$ and $\mathcal{M}$ being constant.

3. Anisotropic Model

In N vortex problem on the plane, vortex filaments are regarded as parpendicular line to the plane and reduced to points on the horizontal plane. In fact, vortex filaments are not linear, nor parpendicular, but wound and inclined more or less. To reflect the inclination on the problem, we introduce anisotropy to the problem.

3.1. One vortex problem

Two vortex problem can be treated as the one vortex problem by taking relative coordinates as well as the Kepler problem. We can easily introduce anisotropy to this problem as follows:

$$H = \gamma \ln(\xi^2 + \varepsilon\eta^2), \qquad (24)$$

where $\varepsilon > 0$ is the coefficient of anisotropy. We can assume $\varepsilon < 1$ without loss of generality. The corresponding Hamiltonian system is reduced to

$$\dot{\xi} = -\frac{\varepsilon\gamma\eta}{\xi^2 + \varepsilon\eta^2}, \qquad \dot{\eta} = \frac{\gamma\xi}{\xi^2 + \varepsilon\eta^2}. \qquad (25)$$

It is easy to show that anisotropic one vortex problem is similar to isotropic one vortex problem.

Theorem: *There exists a scaling transformation which modifies anisotropic one vortex problem to isotropic one vortex problem.*

proof: Introduce the following transformation:

$$Y = \sqrt{\varepsilon}\ \eta, \quad s = \sqrt{\varepsilon}\ t,$$

then, we have

$$\xi' = -\frac{\gamma Y}{\xi^2 + Y^2}, \qquad Y' = \frac{\gamma \xi}{\xi^2 + Y^2},$$

where prime ($'$) denotes derivative with respect to s. □

Using Hamiltonian (22), we can obtain the general solution:

$$(\xi, \eta)(t) = \left(e^{\frac{h}{2\gamma}} \cos(\sqrt{\varepsilon}\ (t - t_0)), \frac{e^{\frac{h}{2\gamma}}}{\sqrt{\varepsilon}} \sin(\sqrt{\varepsilon}\ (t - t_0)) \right), \tag{26}$$

where h is the value of H, and t_0 is the time origin.

4. Further Development of Models

The layered model will be developed to the case of multiple layers with three or more vortex filaments.

The anisotropic model will be developed to the three or more vorteces case.

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