On the external gravitational potential of a three layer Earth model and its influence on the Earth's rotational motion

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Contents

Earth motions

- Classical Celestial Mechanics approach (Tisserand[1] 1892) to model the motion of an extended Earth divides this complex problem in two parts
  - The motion of its barycenter: orbital problem
  - The motion around its barycenter: (mainly) rotational problem
- In general both problems are coupled. However, in the Earth case the orbital problem is almost independent from the rotational one
- In this talk we will focus on certain aspects of the Earth's rotational motion, assuming the orbital motions to be known functions of time

[1] At the end of the document it is given the full information of the references appearing in this work

Earth's rotation

- In addition to its theoretical interest, a precise determination of the rotational motion of the Earth is needed for many practical applications
  - Space navigation
  - Ground–based astrometry
  - Geodesy
  - ...
- Besides, since the rotational motion is affected by the internal structure of our planet, it can also provide insights into the Earth's interior by indirect means

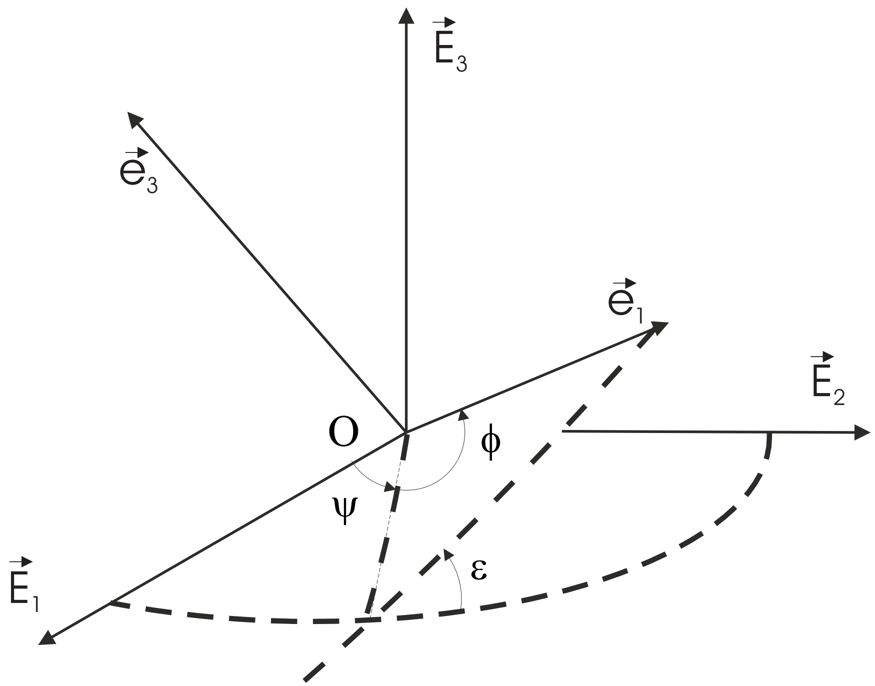
THE EARTH ROTATION PROBLEM

- The Earth's rotation problem consists on determining the operator $\mathsf{R}(t)$ relating celestial and terrestrial systems
- A convenient parameterization of the rotation operator is by means of Euler angles
  $0 \leq \psi(t) < 2\pi,\ 0 < \varepsilon(t) < \pi,\ 0 \leq \phi(t) < 2\pi$

$$\mathsf{R} = \mathsf{R}_{\psi,\varepsilon,\phi} = \mathsf{R}_{\vec{e}_3,\phi}\, \mathsf{R}_{\vec{e}_n,\varepsilon}\, \mathsf{R}_{\vec{E}_3,\psi},$$

where

$$\vec{e}_n = \frac{\vec{E}_3 \times \vec{e}_3}{\|\vec{E}_3 \times \vec{e}_3\|}$$



Form of the solution I

- The particular characteristics of the Earth's rotation makes advisable to separate this motion into three parts: precession–nutation, length of day, and polar motion
- The precession–nutation is related with the evolution of $\vec{e}_3$ (figure axis) in the system $\left\{O; \vec{E}_1, \vec{E}_2, \vec{E}_3\right\}$

$$\vec{e}_3 = \left(\sin\varepsilon(t)\sin\psi\left(t\right), -\sin\varepsilon(t)\cos\psi\left(t\right), \cos\varepsilon(t)\right)^T$$

- Hence, it is provided by determining as functions of time the angles
  - $\psi(t)$, referred as longitude
  - $\varepsilon(t)$, referred as obliquity

Form of the solution II

- Then, it is considered the evolution of the angular velocity vector in the system $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$

$$\mathsf{R}(t)\dot{\mathsf{R}}^T(t) = \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}$$

- It can be expressed in terms of the time derivatives of Euler angles

$$\begin{aligned} \omega_1(t) &= \dot{\psi}(t) \sin\varepsilon(t) \sin\phi(t) + \dot{\varepsilon}(t) \cos\phi(t), \\ \omega_2(t) &= \dot{\psi}(t) \sin\varepsilon(t) \cos\phi(t) - \dot{\varepsilon}(t) \sin\phi(t), \\ \omega_3(t) &= \dot{\psi}(t) \cos\varepsilon(t) + \dot{\phi}(t). \end{aligned}$$

  - $\omega_3(t)$ is related with length of day variations
  - $\omega_1(t)$ and $\omega_2(t)$ define the polar motion (rotational axis)

UNIFORM ROTATION STATE

- Our daily experience shows that the Earth's rotation is not very far from the uniform condition

$$\vec{\omega}(t) = \omega_E \vec{e}_3\,(t)\,,\ \omega_E \sim \frac{360^\circ}{\text{day}}$$

- It implies that
  - There is no polar motion

$$\omega_1(t) = 0,\, \omega_2(t) = 0$$

  - There is no length of day variations

$$\dot{\phi}(t) = \omega_E,\, \phi\,(t) = \omega_E t + \phi_0$$

  - There is no precession–nutation

$$\begin{aligned} \dot{\psi}(t) = 0 \quad &\Rightarrow \quad \psi\,(t) = \psi_0 \sim 0, \\ \dot{\varepsilon}(t) = 0 \quad &\Rightarrow \quad \varepsilon\,(t) = \varepsilon_0 \sim 23.4^\circ \end{aligned}$$

Observed motion

- A careful analysis, however, leads to some small variations with respect to the uniform rotational state
- In this talk we will focus just on precession–nutation changes, although those small variations are also present in polar motion and length of day

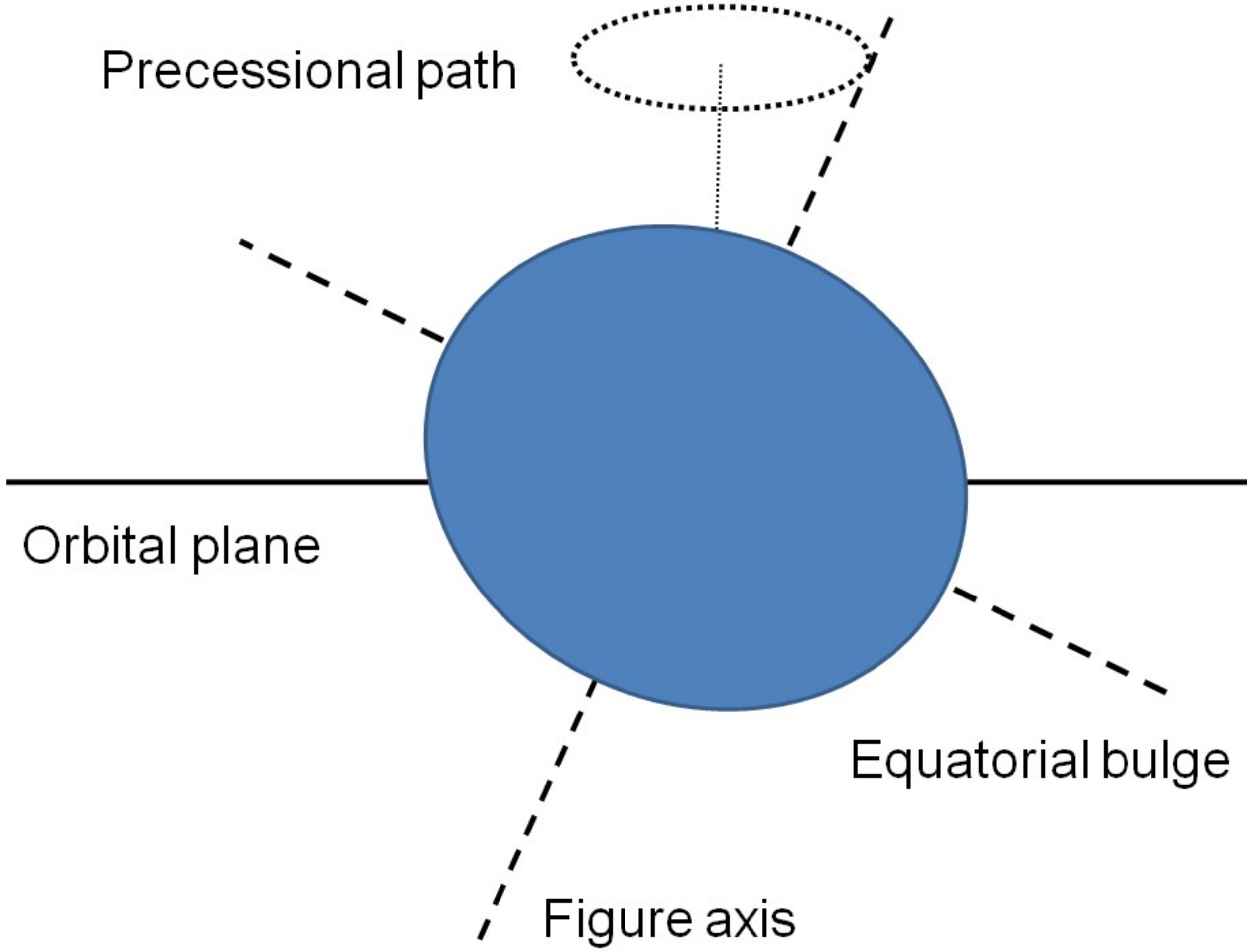
$$\psi(t) \sim \quad 50''t - 17.2''\sin\left(\frac{2\pi}{18.6}t\right) - 1.3''\sin\left(\frac{2\pi}{0.5}t\right) + \ldots$$

$$\varepsilon(t) \sim 23.4^\circ + \quad + 9.2''\cos\left(\frac{2\pi}{18.6}t\right) + 0.6''\cos\left(\frac{2\pi}{0.5}t\right) + \ldots$$

Qualitative explanation I

1. External bodies, mainly the Moon and the Sun, interact gravitationally with the Earth
2. As a consequence of the Earth's equatorial bulge, this interaction creates a torque
3. The torque tries to align the figure axis with the vector normal to the orbital plane
4. Since the Earth is a fast rotator and quasi spherical, this torque creates the weak precession–nutation motion that reflects the periodicity of its cause (Moon and Sun orbital motions)

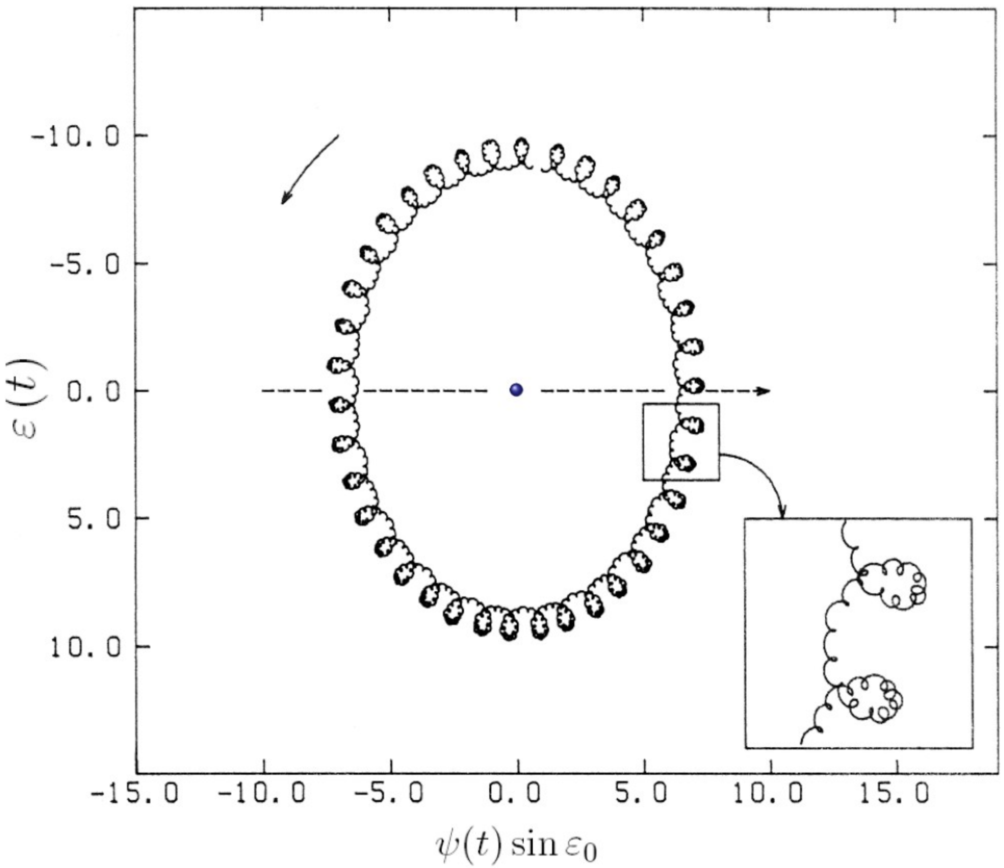
Qualitative explanation II



Representation of the precessional motion (not scaled)

Qualitative explanation III

A closer view into the precessional path shows the nutational motion. For example, over a 18-year period we observe



Taken from Kaplan (2005). Unit= arcsecond

Precession–nutation evolution

- The longitude and the obliquity are given as quasi polynomials

$$\psi(t) = P_\psi(t) + N_\psi(t) + F_\psi(t),\ \varepsilon(t) = P_\varepsilon(t) + N_\varepsilon(t) + F_\varepsilon(t)$$

- $P_l(t)$ is the precession (long periodic motion)

$$P_l(t) = \sum_{j=0} c_j^{(l)} t^j.$$

- $N_l(t)$ is the nutation (quasi periodic motion)

$$N_l(t) = \sum_i a_i^{(l)} \cos\left(n_i t + \Xi_{i0}\right) + b_i^{(l)} \sin\left(n_i t + \Xi_{i0}\right).$$

Orbital frequencies

- In the former expressions $n_i t + \Xi_{i0}$ are related with the orbital motions of the external bodies viewed from Earth
- In the case of the Moon and the Sun, we have

$$n_i t + \Xi_{i0} = \Theta_i = z_{1i} l + z_{2i} l' + z_{3i} F + z_{4i} D + z_{5i} \bar{\Omega},\ z_{ji} \in \mathbb{Z},$$

where l, l', F, D, and $\bar{\Omega}$ are related with the Delaunay variables of the Moon and the Sun

- The particular set of z_{ji} is obtained analyzing the Moon and Sun ephemeris, which are given as functions of time

Current nutation amplitudes

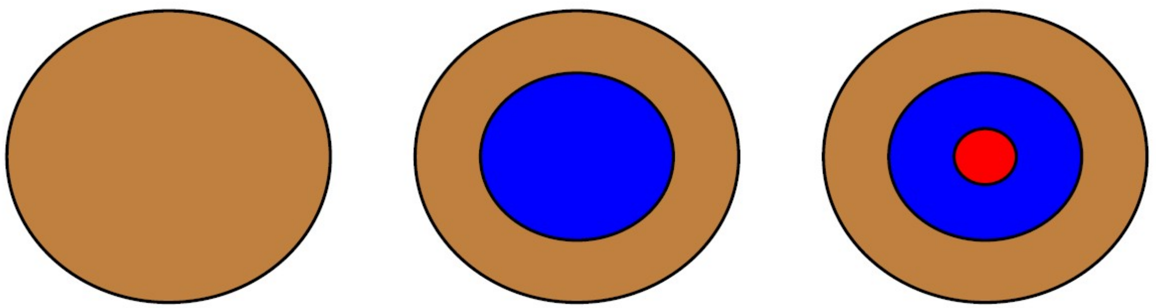
- In particular, the main nutation amplitudes considered nowadays by International Astronomical Union resolutions (2000/2006) are

Argument					Period	Figure axis (arcseconds)	
l_M	l_S	F	D	$\overline{\Omega}$	Days	$\Delta\psi(\sin)$	$\Delta\varepsilon(\cos)$
+0	+0	+0	+0	+1	-6793.48	-17.206416	9.205233
+0	+0	+0	+0	+2	-3396.74	0.207455	-0.089749
+0	+1	+0	+0	+0	365.26	0.147587	0.007387
+0	+0	+2	-2	+2	182.63	-1.317091	0.573034
+0	+1	+2	-2	+2	121.75	-0.051682	0.022439
+0	+0	+2	+0	+2	13.66	-0.227641	0.097846
+0	+0	+2	+0	+1	13.63	-0.038730	0.020073
+1	+0	+2	+0	+2	9.13	0.000082	0.012902

- Note that the amplitudes are given at the level of micro arcsecond
- A detailed explanation of the current standards can be found on IERS Conventions (2010) and Kaplan (2005)

Modeling the Earth's precession–nutation

- In the modeling of the precession–nutation of the Earth enters different aspects, especially considering the nowadays desired level of accuracy of a few micro arcseconds
- One of the most important aspect is the properties of the Earth model under consideration
- In this way, accordingly to the accuracy requirements of each epoch, the complexity of the models considered has been increased
  - 1950's: rigid Earth model
  - 1980's: two layer Earth model
  - 2000's: three layer Earth model



Hamilton's principle formulations

Hamilton's principle can be implemented by

- A Lagrangian function $\mathcal{L} = \mathcal{T} - \mathcal{V}$
  - Holonomic coordinates

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_i}\right) = \mathcal{Q}_i$$

  - Quasi–coordinates

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \omega_i}\right) + \sum_{j,\,k} c_{ijk}\omega_j \frac{\partial \mathcal{L}}{\partial \omega_k} - \sum_{r} \beta_{ri} \frac{\partial \mathcal{L}}{\partial q_r} = \mathcal{Q}_i$$

- A Hamiltonian function $\mathcal{H} = \mathcal{T} + \mathcal{V}$

$$\frac{d}{dt}p_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{Q}_{q_i}, \quad \frac{d}{dt}q_i = \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{Q}_{p_i}$$

Sketch of the procedure

- To describe the evolution of the Earth as a dynamical system we consider that
  - The Earth can be divided in internal quasi spherical layers
  - We consider the motion of all the layers, since it avoids the computations of some internal interactions
  - The velocity in each layer with respect to its Tisserand system (e.g., Escapa 2011) has the form

$$\vec{V}_{(k)} = \vec{\omega}_{(k)} \times \vec{r} + \vec{v}_{d(k)} = \left(\vec{\omega}_E + \delta\vec{\omega}_{(k)}\right) \times \vec{r} + \vec{v}_{d(k)}$$

It is assumed that the rigid part of the field is dominating

$$\left\|\vec{v}_{d(k)}\right\| \sim O(1).$$

Kinetic energy

- The kinetic energy is given by

$$\mathcal{T}_{(k)} = \frac{1}{2}\int_{\mathcal{V}_{(k)}} \left(\vec{V}_{(k)} \cdot \vec{V}_{(k)}\right) \rho_{(k)}(\vec{r})\, d\tau^3$$

- It can be divided into three terms

$$\mathcal{T}_{(k)} = \frac{1}{2}\vec{\omega}_{(k)} \cdot \mathit{\Pi}_k \vec{\omega}_{(k)} + \frac{1}{2}\vec{\omega}_{(k)} \cdot \vec{h}_{(k)} + \frac{1}{2}\int_{\mathcal{V}_{(k)}} \left(\vec{v}_{d(k)} \cdot \vec{v}_{d(k)}\right) \;\rho_{(k)}(\vec{r}) d\tau^3$$

- With the adopted approximations

$$\mathcal{T}_{(k)} = \frac{1}{2}\vec{\omega}_{(k)} \cdot \mathit{\Pi}_k \vec{\omega}_{(k)} = \frac{1}{2}\vec{\omega}_{(k)} \cdot \vec{L}_{(k)}$$

- Therefore, the total kinetic energy of the system

$$\mathcal{T} = \sum_{k=1}^{n} \mathcal{T}_{(k)}$$

Gravitational external potential energy

- Due to the gravitational interaction with external bodies

$$\mathcal{V}_{(k)} = -Gm \int_{\mathcal{V}_{(k)}} \frac{\rho_{(k)}(\vec{r^*})}{\|\vec{r^*} - \vec{r}\|} \, d\tau^{*3}$$

- To work out this expression it is commonly used the expansion (e.g., Kinoshita 1977)

$$\mathcal{V}_{(k)} = -Gm \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \left[\frac{c_{nm(k)}}{r^{n+1}} C_{nm}(\eta, \alpha) + \frac{s_{nm(k)}}{r^{n+1}} S_{nm}(\eta, \alpha) \right],$$

  where C_{ij}, S_{ij} are the real spherical harmonics, and r, η, and α are the distance, colatitude, and longitude of a relevant perturbing body
- The total external potential energy of the Earth is

$$\mathcal{V} = \sum_{p} \sum_{k=1}^{n} \mathcal{V}^{p}_{(k)}$$

Other contributions to the dynamics

- Elastic deformation of the layers
  - Computed through a known expansion of the displacement vector in spheroidal and toroidal harmonics
  - Additional contribution to the kinetic (rotational) energy
  - Additional contribution to the external gravitational potential energy (redistribution tidal potential)
- Dissipative torques
  - Due to electromagnetic and viscous processes
  - Computed through generalized forces
- Internal gravitational potential energy
  - Caused by the gravitational interaction among the Earth layers

Algorithm

Summarizing, in the Hamilton's principle framework

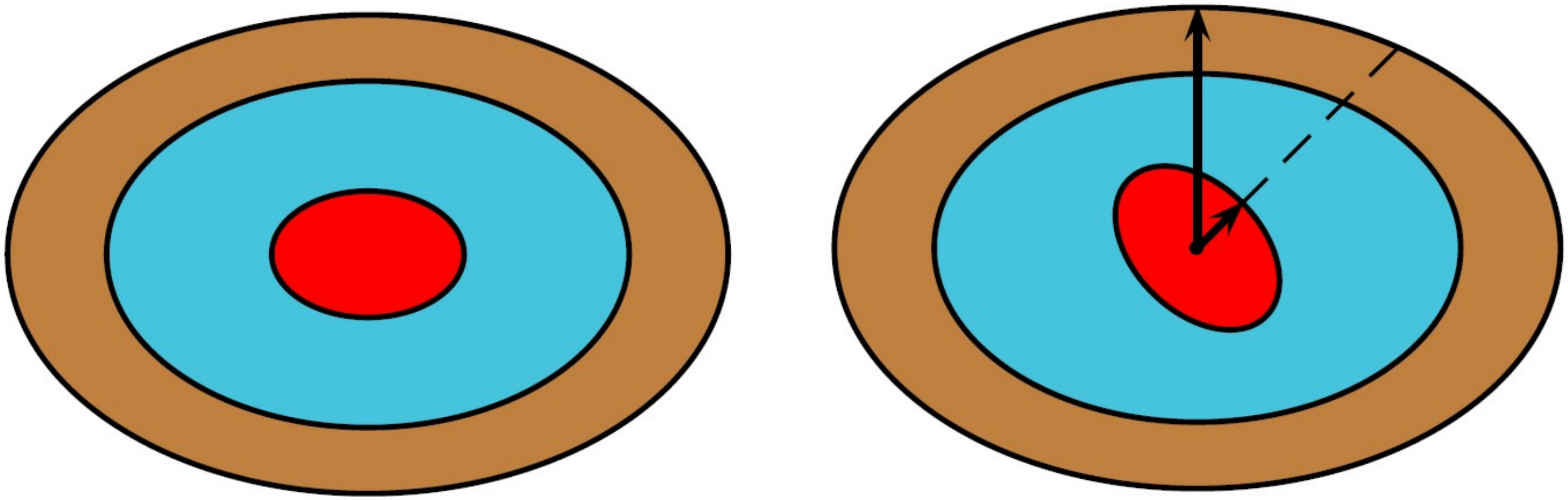
1. The dynamics is described through kinetic, potential energies, and generalized forces
2. The inertia tensors play a central role
   - Increments of inertia tensor have a kinematical or elastic origin

$$\mathit{\Pi}_k = \mathit{\Pi}_{0k} + \Delta_{kin}\mathit{\Pi}_k + \Delta_d\mathit{\Pi}_k$$

   - They induce increments in the kinetic and gravitational potential energy
3. Avoid the computation of pressure torques
4. Similar to the dynamics of several coupled rigid bodies
   - In terms of $\vec{L}_{(k)}$, $\mathit{\Pi}_k$, and $\vec{e}_{i(k)}$.
   - Rheological properties of the Earth are described by a small set of parameters
   - Possible to apply the mathematical tools of Celestial Mechanics

Earth model I

- We will consider an Earth model composed of three nearly spherical, ellipsoidal layers sharing its barycenters
  1. An axial–symmetrical rigid mantle
  2. An fluid outer core
  3. An axial–symmetrical rigid inner core



Earth model II

- Kinematically the configuration of the system is given by
  - Solid layers: defined by rotation matrices $R_{m,s}$, implying

$$\vec{V}_{m,s} = \vec{\omega}_{m,s} \times \vec{r}$$

  - Fluid layer: approximated by a Poincaré flow (e.g., Escapa et al. 2001)

$$\vec{V}_f = \vec{\omega}_f \times \vec{r} \text{ (Tisserand system, } \vec{L}_f = \Pi_f \vec{\omega}_f\text{)}$$

- The main interactions of the system are
  1. Hydrodynamical interaction of the fluid with the solids (internal)
  2. Gravitational perturbations of the Moon and the Sun, whose orbital motion is assumed to be a known function of time (external)

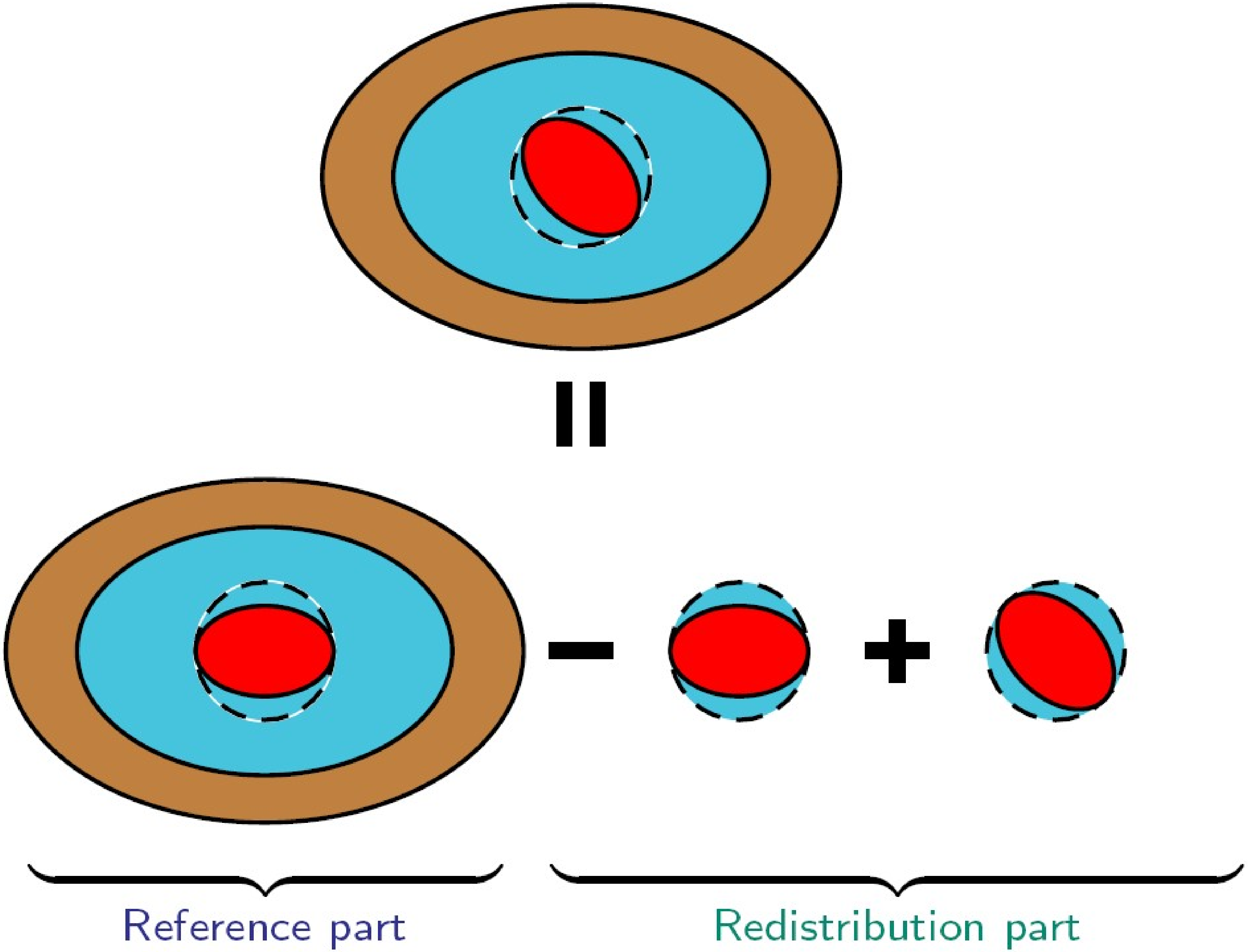
External gravitational potential

- Although there exists a general method for obtaining the expression of the gravitational potential (Escapa et al. 2008), we will focus on the second degree terms since they are responsible of the main contributions to precession–nutation
- Second degree harmonic part of the geopotential can be derived from MacCullagh's formula

$$\mathcal{V} = G\frac{m}{2r^5}\left[3\begin{pmatrix} x \\ y \\ z \end{pmatrix}^t \Pi \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \operatorname{trace}(\Pi)\, r^2\right]$$

- Therefore, to obtain both the reference and the redistribution parts of the potential energy it is necessary to express the matrix of inertia Π of the three layer Earth with respect to a mantle fixed system

Matrix of inertia of the three layer Earth



Matrix of inertia for solid layers

- The mantle and the inner core are assumed to be rigid, so their associated systems are given by the principal axes systems
- In these systems the inertia matrices are
  - Matrix of inertia of the mantle

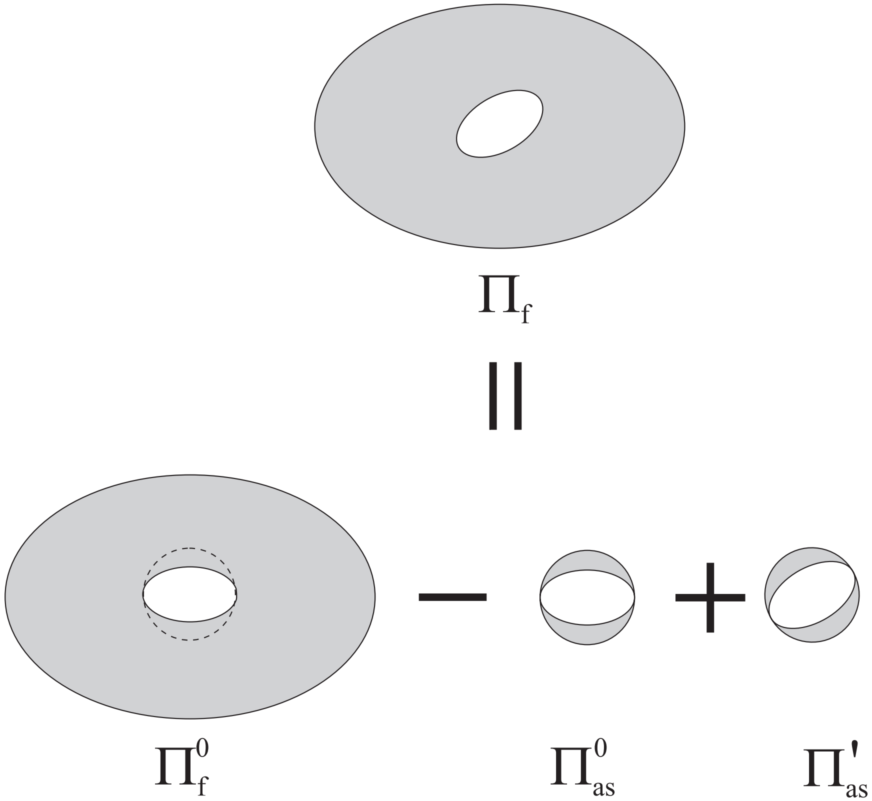
$$\Pi_m = \begin{pmatrix} A_m & 0 & 0 \\ 0 & A_m & 0 \\ 0 & 0 & C_m \end{pmatrix}, \text{ in the } Ox_m y_m z_m \text{ system}$$

  - Matrix of inertia of the inner core

$$\Pi_s = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_s & 0 \\ 0 & 0 & C_s \end{pmatrix}, \text{ in the } Ox_s y_s z_s \text{ system}$$

Matrix of inertia of the fluid layer I

Since the mantle and the inner core evolve independently, view from the mantle the fluid has a time dependent inertia matrix



- We can write

$$\Pi_f = \Pi_f^0 + \Pi'_{as} - \Pi_{as}^0$$

- Π_f^0 and Π_{as}^0 are constant matrices
- Π'_{as} are the time dependent part
- The dependence is entirely due to the relative rotation of the inner core

Matrix of inertia of the fluid layer II

- The matrices of inertia of the fluid and the auxiliary shell (as)

$$\Pi_f^0 = \begin{pmatrix} A_f & 0 & 0 \\ 0 & A_f & 0 \\ 0 & 0 & C_f \end{pmatrix}, \Pi_{as}^0 = \begin{pmatrix} A_{as} & 0 & 0 \\ 0 & A_{as} & 0 \\ 0 & 0 & C_{as} \end{pmatrix}$$

- To compute the matrix Π'_{as}, we have

$$\Pi'_{as} = R_{sm}^T \, \Pi_{as} \, R_{sm}, \quad \text{with } Ox_my_mz_m \xrightarrow{R_{sm}} Ox_sy_sz_s$$

- Then, the inertia matrix of the fluid is

$$\Pi_f = \begin{pmatrix} A_f & 0 & 0 \\ 0 & A_f & 0 \\ 0 & 0 & C_f \end{pmatrix} + (C_{as} - A_{as}) \begin{pmatrix} k_1^2 & k_1k_2 & k_1k_3 \\ k_1k_2 & k_2^2 & k_2k_3 \\ k_1k_3 & k_2k_3 & k_3^2 - 1 \end{pmatrix},$$

k_i being the components of the inner core figure axis

Matrix of inertia of the Earth: final form

- The matrix of inertia is the sum of those of its constituents

$$\Pi = \Pi_m + \Pi_f + \Pi_s$$

- Accordingly to the previous results we can split it in the form

$$\Pi = \left(\Pi_m + \Pi_f^0 + \Pi_s^0\right) + \left(\Pi_s' - \Pi_s^0 + \Pi_{as}' - \Pi_{as}^0\right)$$

- Explicitly,

$$\Pi = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+e \end{pmatrix} + A_s\left(e_s - \delta\right) \begin{pmatrix} k_1^2 & k_1 k_2 & k_1 k_3 \\ k_1 k_2 & k_2^2 & k_2 k_3 \\ k_1 k_3 & k_2 k_3 & k_3^2 - 1 \end{pmatrix},$$

with

$$\begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix}^T = R_{sm}^T \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T,$$

and the ellipticities

$$e = \frac{C - A}{A}, \; e_s = \frac{C_s - A_s}{A_s}, \; \delta = \frac{A_{as} - C_{as}}{A_{as}}$$

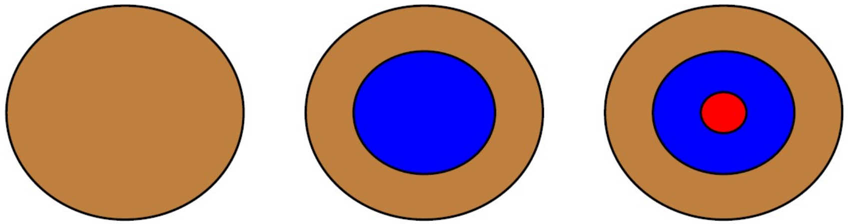
External gravitational potential expression

- Therefore, the external gravitational potential can be written as (Escapa et al. 2011, 2012)

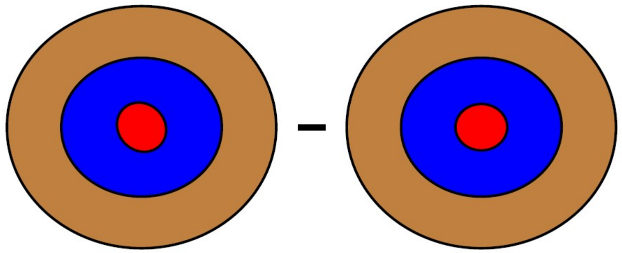
$$
\begin{aligned}
\mathcal{V} = &\quad \mathcal{V}_2 + \Delta\mathcal{V} = \\
= &\quad G\frac{m}{r^3}\left\{AeC_{20}\left(\eta\right) + \right. \\
+ &\quad A_s\left(e_s - \delta\right)\left[\frac{2\left(k_3^2 - 1\right) - k_1^2 - k_2^2}{2}C_{20}\left(\eta, \alpha\right) + k_1k_3C_{21}\left(\eta, \alpha\right) + \right. \\
+ &\quad \left.\left. k_2k_3S_{21}\left(\eta, \alpha\right) + \frac{k_1^2 - k_2^2}{4}C_{22}\left(\eta, \alpha\right) + \frac{k_1k_2}{2}S_{22}\left(\eta, \alpha\right)\right]\right\}
\end{aligned}
$$

External gravitational potential expression

- The term $\mathcal{V}_2$ is common for one, two, and three layer Earth models, since depends on the moments of inertia of the whole Earth (although the response to it is different)



- The term $\Delta\mathcal{V}$ is intrinsically due to the three layer Earth model. It is originated by the differential rotation of the inner core with respect to the mantle



- As far as we know, its effects on the Earth's precession–nutation has not been quantified previously

Hamiltonian formalism

Earth rotation studies with Hamiltonian formalism (some examples)

1. One layer rigid Earth models
   - Kinoshita (1977)
   - Souchay, Losley, Kinoshita, & Folgueira (1999)
   - Escapa, Getino, & Ferrándiz (2002)
   - Getino, Escapa, & Miguel (2010)
2. One layer elastic Earth models
   - Kubo (1991, 2009)
   - Getino & Ferrándiz (1995)
   - Escapa (2011)
3. Two layer Earth models
   - Kubo (1979)
   - Getino (1995a, 1995b)
   - Getino & Ferrándiz (1997, 1999, 2000, 2001)
   - Ferrándiz, Navarro, Escapa, & Getino (2004)
4. Three layer Earth models
   - Escapa, Getino, & Ferrándiz (2001, 2011)

Hamilton equations

The evolution of a dynamical system is described in terms of a set of canonical variables $(p,\, q)$ and Hamilton equations

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + \mathcal{Q}_{q_i},\; \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{Q}_{p_i},$$

where the Hamiltonian can be written as

$$\mathcal{H} = \mathcal{T} + \mathcal{V},$$

and $\mathcal{Q}_p$, $\mathcal{Q}_q$ are the canonical forces, accounting for dissipative processes

Canonical variables: Andoyer variables

- The moments p are given by

$$p_1 = \vec{L} \cdot \vec{E}_3,\ p_2 = \vec{L} \cdot \vec{e}_{\vec{L}},\ p_3 = \vec{L} \cdot \vec{e}_3$$

- The conjugate variables q are define with the help of

$$\vec{e}_I = \frac{\vec{E}_3 \times \vec{e}_{\vec{L}}}{\|\vec{E}_3 \times \vec{e}_{\vec{L}}\|},\ \vec{e}_\sigma = \frac{\vec{e}_{\vec{L}} \times \vec{e}_3}{\|\vec{e}_{\vec{L}} \times \vec{e}_3\|}$$

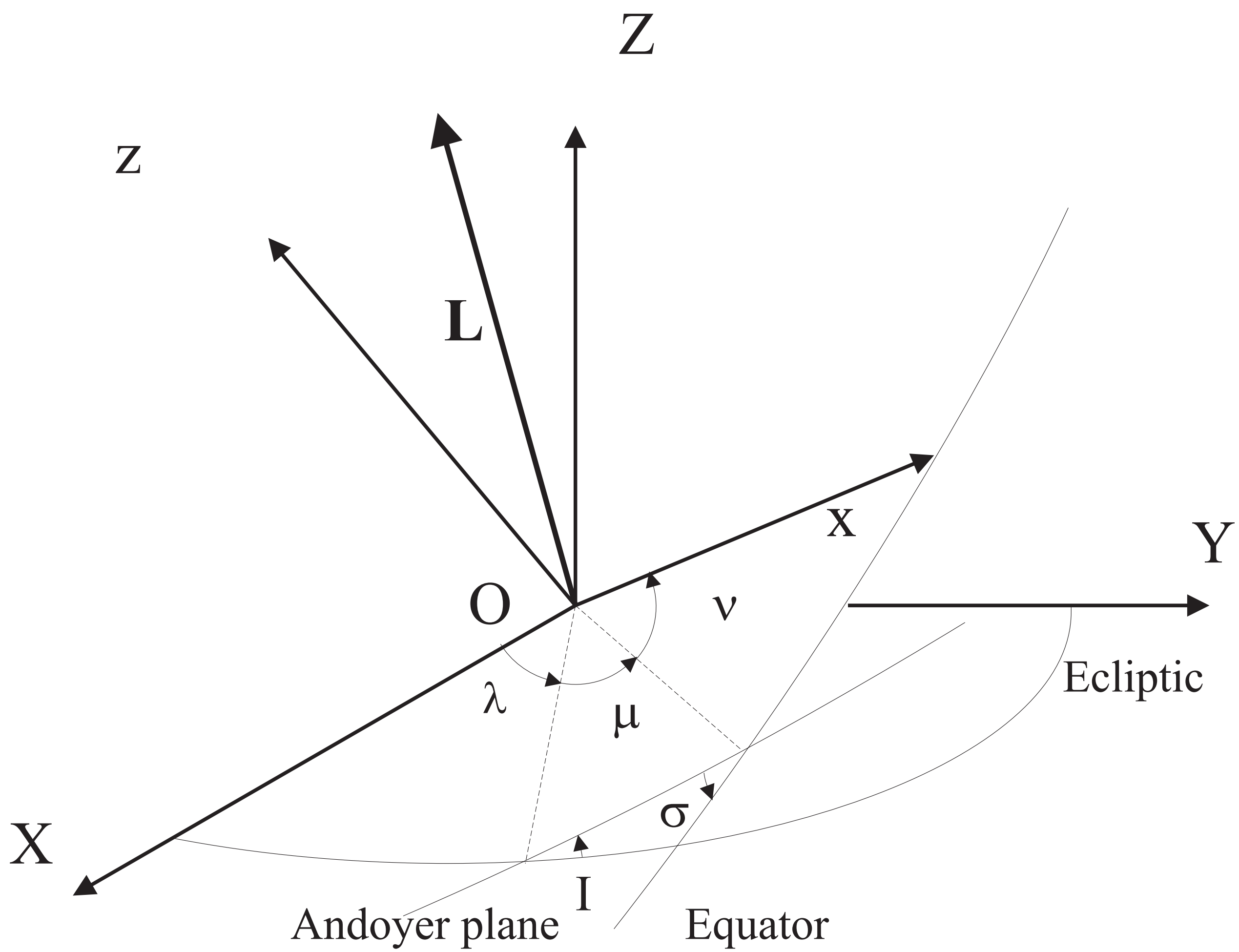
- We have that

$$q_1 \to \vec{E}_3 \angle \vec{e}_I,\ q_2 \to \vec{e}_I \angle \vec{e}_\sigma,\ q_3 \to \vec{e}_\sigma \angle \vec{e}_3$$

- Usually, we note the Andoyer canonical set as

$$p_2 = M,\ p_1 = \Lambda = M\ \cos I,\ p_3 = N = M\ \cos\sigma,$$
$$q_1 = \lambda,\ q_2 = \mu,\ q_3 = \nu$$

Canonical variables: Andoyer variables



Canonical variables: Andoyer modified set

- Variables associated to the mantle

$$\begin{pmatrix} x_{1(m)} \\ x_{2(m)} \\ x_{3(m)} \end{pmatrix} = R_3(\nu)R_1(\sigma)R_3(\mu)R_1(I)R_3(\lambda) \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix},$$

$$\Lambda = \vec{L} \cdot \vec{E}_3, \; M = \vec{L} \cdot \vec{e}_{\vec{L}}, \; N = \vec{L} \cdot \vec{e}_{3(m)}$$

- Variables associated to the remaining layers

$$\begin{pmatrix} x_{1(k)} \\ x_{2(k)} \\ x_{3(k)} \end{pmatrix} = R_3(\lambda_{(k)})R_1(I_{(k)})R_3(\mu_{(k)})R_1(\sigma_{(k)})R_3(\nu_{(k)}) \begin{pmatrix} x_{1(m)} \\ x_{2(m)} \\ x_{3(m)} \end{pmatrix},$$

$$N_{(k)} = \vec{L}_{(k)} \cdot \vec{e}_{3(m)}, \; M_{(k)} = \vec{L}_{(k)} \cdot \vec{e}_{\vec{L}_{(k)}}, \; \Lambda_{(k)} = \vec{L}_{(k)} \cdot \vec{e}_{3(k)}$$

Andoyer modified set: dynamical meaning

We have the following relationships

- Angular momentum in the terrestrial frame

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} M \sin\sigma \sin\nu \\ M \sin\sigma \cos\nu \\ N \end{pmatrix}, \quad \begin{pmatrix} L_{1(k)} \\ L_{2(k)} \\ L_{3(k)} \end{pmatrix} = \begin{pmatrix} M_{(k)} \sin\sigma_{(k)} \sin\nu_{(k)} \\ -M_{(k)} \sin\sigma_{(k)} \cos\nu_{(k)} \\ N_{(k)} \end{pmatrix}$$

- Angular velocity

$$\vec{\omega}_{(k)} = \Pi_k^{-1} \cdot \vec{L}_{(k)}, \; \vec{\omega}_{(m)} = \Pi_m^{-1} \cdot (\vec{L} - \sum_{k \neq m} \vec{L}_{(k)})$$

- Relation with Euler angles

$$\psi = \lambda + \sigma \frac{\sin\mu}{\sin I}, \; \varepsilon = I + \sigma\cos\mu, \; \phi = \mu + \nu - \sigma \frac{\cos I \sin\mu}{\sin I}$$

Approximate analytical solution

- Direct integration of equations of motion is unfeasible

$$\frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}, \ \frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}$$

- The Hamiltonian of our system always can be decomposed

$$\mathcal{H} = \mathcal{H}_0 + \chi \mathcal{H}_1, \ \text{with } 0 \leq \chi \ll 1$$

- To use perturbation methods requieres a known solution of

$$\frac{dp_i}{dt} = -\frac{\partial \mathcal{H}_0}{\partial q_i}, \ \frac{dq_i}{dt} = \frac{\partial \mathcal{H}_0}{\partial p_i}$$

- In the Earth case

$$\frac{|\mathcal{H} - \mathcal{H}_0|}{|\mathcal{H}_0|} \sim 10^{-7},$$

where, at least, $\mathcal{H}_0$ must contain the main part of $\mathcal{T}$

Hori's method (Hori 1966): first order

- To perform a canonical transformation $(p, q) \rightarrow (p^*, q^*)$

$$\mathcal{K}(p^*, q^*) = \mathcal{K}_0 + \mathcal{K}_1, \; \mathcal{W}(p^*, q^*) = \mathcal{W}_1$$

- It can be achieved using an average condition

$$\mathcal{K}(p^*, q^*) = \mathcal{H}_0 + \mathcal{H}_{1\text{sec}}, \; \mathcal{W}(p^*, q^*) = \int_{\text{UP}} \mathcal{H}_{1\text{per}} \, dt$$

- The integral to obtain $\mathcal{W}$ is computed over the unperturbed solutions

$$\frac{dp_i}{dt} = -\frac{\partial \mathcal{H}_0}{\partial q_i}, \; \frac{dq_i}{dt} = \frac{\partial \mathcal{H}_0}{\partial p_i}$$

- The evolution of any function is given by through

$$f(p, q) = f(p^*, q^*) + \{f, \mathcal{W}\},$$

being (p^*, q^*) the solution of the transformed system

Canonical formulation of our problem

- In the case of the three layer problem under consideration
  - We express the kinetic and potential energy in terms of an Andoyer modified set

$$\vec{L} = \begin{pmatrix} M \sin\sigma \sin\nu \\ M \sin\sigma \cos\nu \\ N \end{pmatrix}, \vec{L}_{f,\,s} = \begin{pmatrix} M_{f,\,s} \sin\sigma_{f,\,s} \sin\nu_{f,\,s} \\ -M_{f,\,s} \sin\sigma_{f,\,s} \cos\nu_{f,\,s} \\ N_{f,\,s} \end{pmatrix},$$

$$R_{sm} = R_3\left(\lambda_s\right) R_1\left(I_s\right) R_3\left(\mu_s\right) R_1\left(\sigma_s\right) R_3\left(\nu_s\right)$$

  - At the first order, we can take the Hamiltonian as

$$\mathcal{H} = \mathcal{T} + \Delta\mathcal{V}$$

  - The nutations come from $\Delta\mathcal{V}$, to be determined with Hori's method by taking $\mathcal{H}_0 = \mathcal{T}$.

Kinetic energy

- We assume a rigid rotation field of velocities
- When the kinematical increment of the inertia tensor is small

$$(\Pi_f)^{-1} = (\Pi_f^0)^{-1} - (\Pi_f^0)^{-1} (\Pi'_{as} - \Pi_{as}^0) (\Pi_f^0)^{-1}$$

- The kinetic energy of the system can be written

$$\begin{aligned}\mathcal{T} \;=\; & \frac{1}{2}\left(\vec{L} - \vec{L}_f - \vec{L}_s\right)^T \Pi_m^{-1} \left(\vec{L} - \vec{L}_f - \vec{L}_s\right) \\ & + \frac{1}{2}\vec{L}_f^T \, (\Pi_f^0)^{-1} \, \vec{L}_f + \frac{1}{2}\vec{L}_s^T \, \Pi_s^{-1} \, \vec{L}_s \\ & - \frac{1}{2}\vec{L}_f^T \left[(\Pi_f^0)^{-1} (\Pi'_{as} - \Pi_{as}^0) (\Pi_f^0)^{-1}\right] \vec{L}_f\end{aligned}$$

Kinetic energy in Andoyer variables

- The kinetic energy of the system is (Escapa et al. 2001)

$$\begin{aligned}
\mathcal{T} \;=\; & \frac{1}{2A_m}\left(K^2 + \frac{A_m + A_f}{A_f}K_f^2 + K_s^2\right) + \frac{1}{2C_m}\left(N^2\right. \\
& \left. + \frac{C_m + C_f}{C_f}N_f^2 + N_s^2 - 2NN_f - 2NN_s + 2N_fN_s\right) \\
& + \frac{KK_f}{A_m}\cos\left(\nu + \nu_f\right) + \frac{KK_s}{A_m}\cos\left(\nu + \nu_s\right) \\
& + \frac{K_fK_s}{A_m}\cos\left(\nu_f - \nu_s\right) + \frac{1}{2}\left(\frac{1}{C_s} - \frac{1}{A_s}\right)\Lambda_s^2 \\
& + \frac{1}{2A_s}M_s^2 + \mathcal{T}_{as},
\end{aligned}$$

  where

$$K = \sqrt{M^2 - N^2},\; K_{f,s} = \sqrt{M_{f,s}^2 - N_{f,s}^2}$$

- $\mathcal{T}_{as}$ is the term responsible for the coupling between mantle and the inner core through fluid interaction

Potential energy in canonical variables

- The potential energy $\Delta\mathcal{V}$ is given by

$$\Delta\mathcal{V} = A_s\left(e_s - \delta\right)\left[\frac{2\left(k_3^2-1\right)-k_1^2-k_2^2}{2}C_{20}\left(\eta,\alpha\right)+k_1k_3C_{21}\left(\eta,\alpha\right)+\right.$$

$$\left.+\quad k_2k_3S_{21}\left(\eta,\alpha\right)+\frac{k_1^2-k_2^2}{4}C_{22}\left(\eta,\alpha\right)+\frac{k_1k_2}{2}S_{22}\left(\eta,\alpha\right)\right]\Bigg\}$$

- We must to express it in terms of the canonical variables
  - The rotation R_{sm} allows us to write k_i

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = R_{sm}^T\left(\lambda_s, I_s, \mu_s, \sigma_s, \nu_s\right)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  - The geocentrical coordinates (r,η,α) must be referred to the ecliptic of date (Kinoshita 1977)

Analytical solution

- The Hamiltonian is now expressed in an Andoyer–like canonical set of variables
$$\mathcal{H} = \mathcal{T} + \Delta\mathcal{V}$$
- Since the direct solution of the equations of motion is not possible, we will apply Hori's perturbation method
- The unperturbed part $\mathcal{H}_0 = \mathcal{T}$ accounts for the hydrodynamical internal interactions
- The perturbation term $\mathcal{H}_1 = \Delta\mathcal{V}$ is due to the change in the external gravitational potential caused by the relative rotation of the inner core
- This procedure allows us to obtain first order analytical approximate solutions to the contribution to the Earth's precession–nutation

Unperturbed problem

- The equations of motion are given by

$$\dot{p} = -\frac{\partial \mathcal{T}}{\partial q}, \ \dot{q} = \frac{\partial \mathcal{T}}{\partial p}$$

- The evolution of some canonical variables is direct

$$M = C\omega_E, \ \Lambda = \text{cte.} \ \lambda = \text{cte.}, \ \mu + \nu = \omega_E t + \omega_{E0}$$

- The time evolution of the remaining variables is given by the system

$$\begin{pmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \\ \dot{Z} \end{pmatrix} = i\omega_E J \begin{pmatrix} U \\ V \\ W \\ Z \end{pmatrix}, \text{ where } \begin{cases} U = iM \sin\sigma e^{-i\nu} \\ V = -iM_f \sin\sigma_f e^{-i\nu_f} \\ W = -iM_s \sin\sigma_s e^{-i\nu_s} \\ Z = iM_s \sin I_s e^{i(\mu_s+\nu_f)} \end{cases}$$

Generating function. First order solution

- The secular part of the potential is given by the average over the unperturbed problem

$$\mathcal{H}_{1\mathsf{sec}} = < \Delta\mathcal{V} > = 0,$$

since all the perturbation is (quasi) periodic

- The periodic part provides the generating function

$$\mathcal{W} = \int_{\mathsf{UP}} \Delta\mathcal{V}\, dt$$

- Since $\mathcal{H}_{1\mathsf{sec}} = 0$, there is no contribution to the precession
- With respect to the nutation, we have that
  - The nutation in longitude is computed through

$$\Delta\psi = \{\psi, \mathcal{W}\}$$

  - The nutation in obliquity is computed through

$$\Delta\varepsilon = \{\varepsilon, \mathcal{W}\}$$

Numerical estimation

The order of magnitude of the contributions previously determined analytically (Escapa et al. 2011, 2012) is

Argument					Period	Figure axis (μas)	
l_M	l_S	F	D	$\overline{\Omega}$	Days	$\Delta\psi$	$\Delta\varepsilon$
+0	+0	+0	+0	+1	-6793.48	2.79	-0.31
+0	+0	+0	+0	+2	-3396.74	0.00	-0.01
+0	+1	+0	+0	+0	365.26	14.95	9.29
+0	-1	+2	-2	+2	365.25	-1.78	0.48
+0	+0	+2	-2	+2	182.63	44.61	-19.92
+0	+1	+2	-2	+2	121.75	1.64	-0.72
+1	+0	+0	+0	+0	27.55	-2.22	0.02
+0	+0	+2	+0	+2	13.66	7.17	-3.08
+0	+0	+2	+0	+1	13.63	1.22	-0.63
+1	+0	+2	+0	+2	9.13	0.96	-0.41

- The presence of the inner core induces a new contribution into the external gravitational potential of the Earth
- This contribution is intrinsically a three layer effect, not present in one or two layer models, due to the relative rotation of the inner core with respect to the mantle
- By means of a Hamiltonian approach we have obtained the effect of this variation on the rotation of the Earth (precession–nutation)
- Specifically, the motion of the figure axis is affected through new contributions to the nutational terms
- The amplitudes of the new contributions are of the order of tens (μas) for some nutational arguments
- As far as we know, these contributions are not taken into account currently. In view of its magnitude they should be incorporated to the actual standards and models

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