Variational construction of some satellite orbits

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Satellite orbits

In a periodic gravitational force field, we call the orbit of a zero (negligible) mass a satellite orbit.

Motivating examples:

1. Restricted n-body problem

2. Spacecraft near an asteroid

Mathematical theories for the existence of various periodic satellite orbits: analytic continuation, power series method, equating Fourier coefficients, fixed point methods.
[V. Szebehely: *Theory of orbits – the restricted 3-body problem*. 1967]

How about variational methods?

Restricted n-body problem

Let $x_k \in \mathbb{R}^d$ be the position of mass $m_k > 0$.
Equations of motions for the n-body problem (NBD):

$$m_k \ddot{x}_k = \sum_{i \neq k} \frac{m_i m_k (x_i - x_k)}{|x_i - x_k|^3}, \qquad k = 1, \ldots, n.$$

The special case n = 2 is also called the Kepler problem.

When there is a zero mass, it is called the restricted n-body problem (RNBD). Usually we consider the case nonzero masses (primaries) forming a periodic solution. By satellite orbit we mean the orbit of some zero mass.

Restricted n-body problem

Equations of motions for the Kepler problem:

$$\ddot{x}_i = \frac{m_j(x_j - x_i)}{|x_j - x_i|^3}, \quad \{i, j\} = \{1, 2\}.$$

Given a elliptical solution for the Kepler problem.
The equation of motion for the (R3BD) is

$$\ddot{q} = \frac{m_1(x_1 - q)}{|x_1 - q|^3} + \frac{m_2(x_2 - q)}{|x_2 - q|^3}$$

Restricted n-body problem

Consider the Kepler problem with masses $m_1 = 1 - \mu,\ m_2 = \mu$ and a circular Keplerian orbit

$$x_1(t) = \mu e^{it}, \quad x_2(t) = -(1-\mu)e^{it}$$

The equation of motion for the circular restricted 3-body problem (CR3BD) is

$$\ddot{q} = \frac{m_1(x_1 - q)}{|x_1 - q|^3} + \frac{m_2(x_2 - q)}{|x_2 - q|^3} = \frac{\partial}{\partial q} U(q,t),$$

where

$$U(q,t) = \frac{1-\mu}{\sqrt{|q|^2 + \mu^2 - 2\mu|q|\cos(\theta - t)}} + \frac{\mu}{\sqrt{|q|^2 + (1-\mu)^2 + 2(1-\mu)|q|\cos(\theta - t)}}.$$

Restricted n-body problem

Considering the rotating coordinate system

$$z = x + iy = e^{-it}q.$$

Then the equation of motion becomes

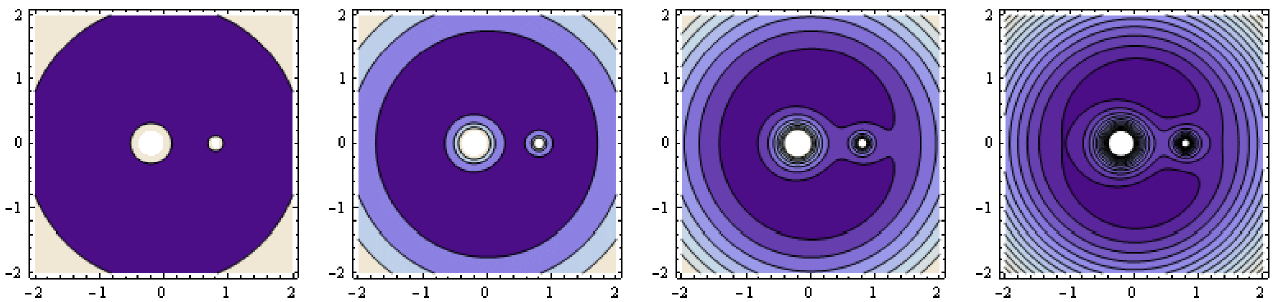
$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial V}{\partial x}\\ \ddot{y} + 2\dot{x} &= \frac{\partial V}{\partial y}\end{aligned}$$

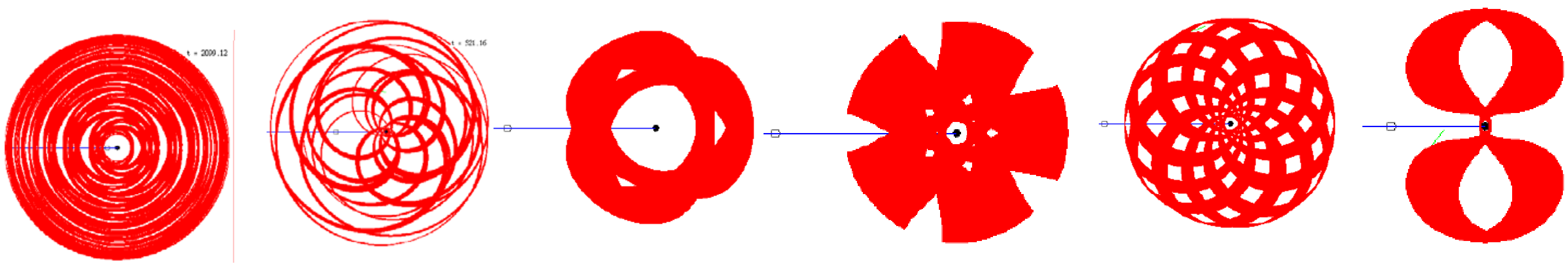
where

$$V(x,y) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{\sqrt{(x-\mu)^2+y^2}} + \frac{\mu}{\sqrt{(x-1+\mu)^2+y^2}}$$

Restricted n-body problem

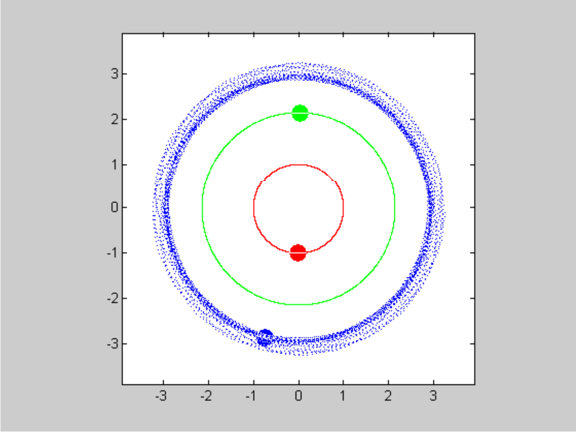
There is a Jacobi integral $h = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y)$
which confines motion of the satellite in certain regions (Hill region). Boundaries for them are level curves of *V*.

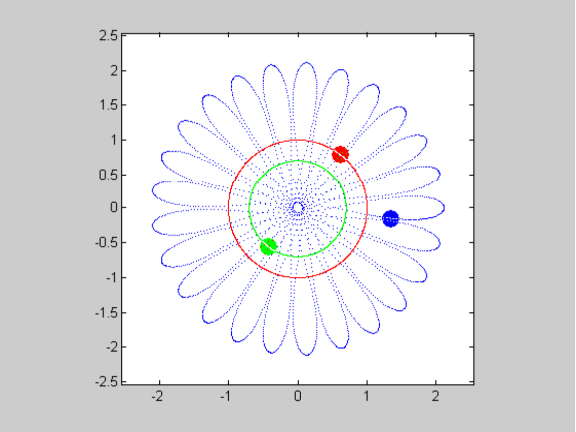




Restricted n-body problem

Examples of retrograde solutions for (3BD) with one very small (blue) mass, proved to exist in [K.C. *Ann. Math.* 2008]:





The proof for the existence of such orbits also motivates our study of (RNBD).

Restricted n-body problem

Equations of motions for the 3-body problem:

$$\ddot{x}_i = \frac{m_j(x_j - x_i)}{|x_j - x_i|^3} + \frac{m_k(x_k - x_i)}{|x_k - x_i|^3}, \quad \{i, j, k\} = \{1, 2, 3\}.$$

Given a periodic solution for the 3-body problem.
The equation of motion for the (R4BD) is

$$\ddot{q} = \frac{m_1(x_1 - q)}{|x_1 - q|^3} + \frac{m_2(x_2 - q)}{|x_2 - q|^3} + \frac{m_3(x_3 - q)}{|x_3 - q|^3}$$

which can be formulated and studied as (R3BD).

Spacecraft near an asteroid

A model for spacecraft motion about an elliptical asteroid which rotates with unit angular velocity:

$$\ddot{x} - 2\dot{y} = \frac{\partial V}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial V}{\partial y},$$

where the potential V takes the form

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{\sqrt{x^2 + y^2}} + V_{20}(x, y) + V_{22}(x, y).$$

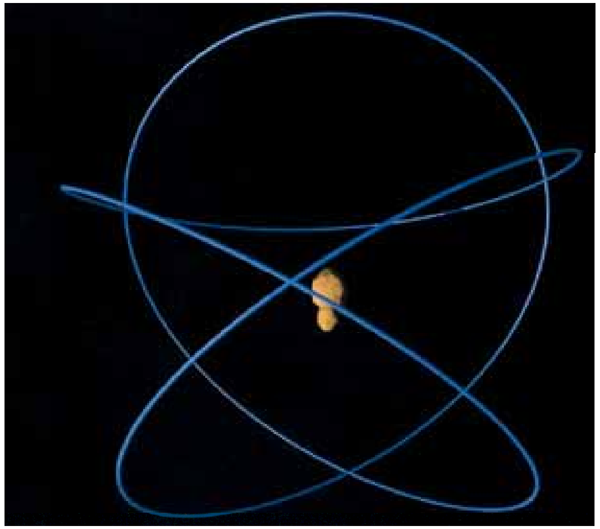
The last two terms are small order terms due to ellipticity.

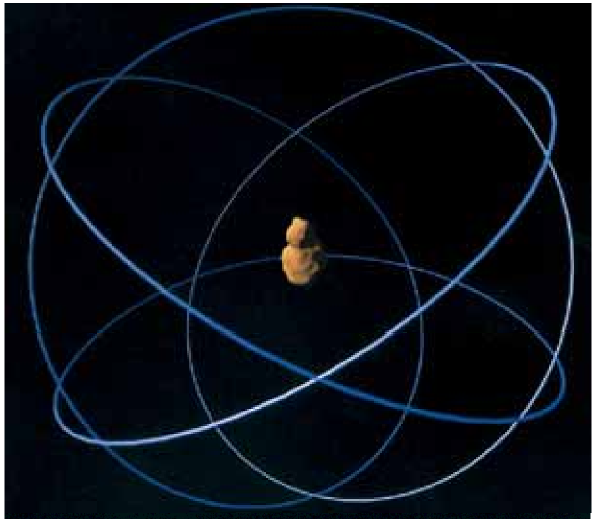
[Koon-Marsden-Ross-Lo-Scheeres: Geometric mechanics and the dynamics of asteroid pairs, *Ann. N.Y. Acad. Sci, 2004*]
(survey + refs)

Spacecraft near an asteroid

Near a more irregular asteroid, the dynamics near it are often more abundant.

Borrowing some numerical figures from [Scheeres-Ostro-Hudson-DeJong-Suzuki, *Icarus*, 1998]





Variational setting

A common feature of these problems is that their equations of motions can be expressed

(SO) $$\ddot{q} = \frac{\partial}{\partial q} U(q,t),$$

where U is a time-periodic potential. Let total mass be 1. Then

$$U(q,t) = \frac{1}{|q|} + U_0(q,t)$$

where U_0 is the perturbation from point-mass potential.

The perturbation can be measured by

$$R_\rho = \sup_{|q| \geq \rho, t \in \mathbb{R}} |U_0(q,t)|$$

Variational setting

Roughly speaking, if the zero mass is not very close to the primaries, then (SO) is close to the Kepler problem.

For (RNBD), regard the aggregate of primaries as one celestial body.

The kind of 2-body problem described by (SO) is often called a restricted full 2-body problem.

Variational setting

The equation (SO) is the Euler-Lagrange equation for

$$\mathcal{A}_T(q) = \int_0^T L(q, \dot{q}, t)dt, \quad L(q, \dot{q}, t) = \frac{1}{2}|\dot{q}|^2 + U(q, t).$$

Consider a "natural" function space for the action functional: the Sobolev space $H = H^1([0, T], \mathbb{C})$ and a "punctured" subspace:

$$H_\rho = \{x \in H^1([0, T], \mathbb{C}) \colon |x(t)| \geq \rho \text{ for all } t\}.$$

Our goal is to characterize satellite orbits as critical points of the action functional in the interior of some punctured subspace of H.

Variational setting

Assumptions on the perturbation U_0 from point-mass potential:

(A_1) (Regularity) For some $\rho_0 \in (0, \rho)$, $U_0 \in C^0((\mathbb{C} \setminus D_{\rho_0}) \times \mathbb{R})$ is locally Lipschitz in q and $|U_0(q,t)|\,|q| \to 0$ as $|q| \to \infty$.

(A_2) (Periodicity) For any $q \in \mathbb{C}^\times$, $U_0(e^{it}q, t)$ is independent of t.

Consider relative T-periodic loops

$$\Lambda_T = \{q \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{C}) \colon\ e^{-iT} q(t+T) = q(t) \text{ for any } t\}$$

and the restriction to some punctured subspace of H

$$\Lambda_{T,\rho} = \{q \in \Lambda_T \colon\ q\,|_{[0,T]} \in H_\rho\}$$

Assume T is not equal to $2k\pi,\ k \in \mathbb{N}$.

Main theorems

Proposition. The action functional restricted to $\Lambda_{T,\rho}$ attains its infimum.

Define $\kappa(T,\rho) = \min_{s\in(0,\infty)} \left\{ \frac{1}{2T^2}(s-\rho)^2 + \frac{1}{s} \right\}.$

The function $\kappa(T,\rho)$ has a closed form:

$$\alpha = \alpha(T,\rho) = 2^{-\frac{1}{3}} \left(2\rho^3 + 27T^2 + 3\sqrt{3}T\sqrt{4\rho^3 + 27T^2} \right)^{\frac{1}{3}}$$

$$s_{\min} = \frac{1}{3}\left(\rho + \frac{\rho^2}{\alpha} + \alpha \right)$$

$$\kappa(T,\rho) = \frac{1}{2T^2}(s_{\min} - \rho)^2 + \frac{1}{s_{\min}}.$$

Main theorems

Theorem1. (K.C. *Amer. J. Math.* 2010)

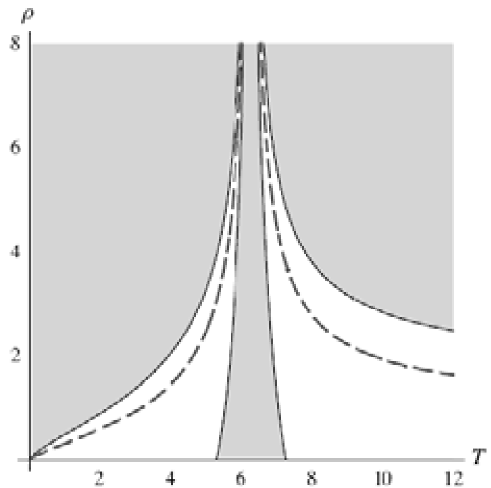
Suppose $\rho \leq \left|1 - \frac{2\pi}{T}\right|^{-\frac{2}{3}}$,

(*) $$\frac{3}{2}\left|1-\frac{2\pi}{T}\right|^{\frac{2}{3}} + 2R_\rho < \kappa(T,\rho) + \frac{\rho^2}{2}\left|1-\frac{2\pi}{T}\right|^2,$$

then there exists a minimizing classical solution for (SO) in the interior of $\Lambda_{T,\rho}$. The satellite orbit is retrograde if $T \in (0, 2\pi)$, direct if $T > 2\pi$.

Regions on which

$$\frac{3}{2}\left|1-\frac{2\pi}{T}\right|^{\frac{2}{3}} < \kappa(T,\rho) + \frac{\rho^2}{2}\left|1-\frac{2\pi}{T}\right|^2$$



Main theorems

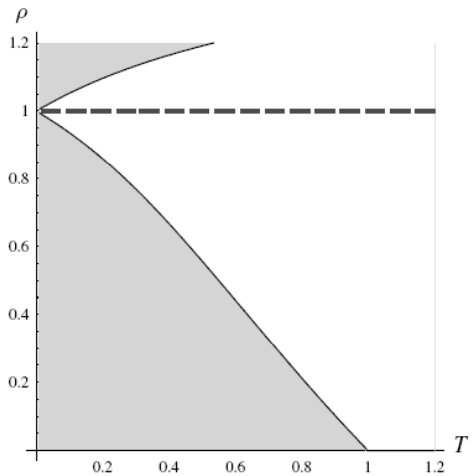
Theorem2. (K.C. *Amer. J. Math.* 2010)
Suppose $\rho \leq 1$,

(**) $$\frac{3}{2} + 2R_\rho < \kappa(T,\rho) + \frac{\rho^2}{2},$$

then there exists a minimizing classical direct solution for (SO) in the interior of $\Lambda_{T,\rho}$.

Regions on which

$$\frac{3}{2} < \kappa(T,\rho) + \frac{\rho^2}{2}$$



Revisit: (CR3BD)

Consider the Kepler problem with masses $m_1 = 1 - \mu$, $m_2 = \mu$ and a circular Keplerian orbit

$$x_1(t) = \mu e^{it}, \quad x_2(t) = -(1-\mu)e^{it} \quad .$$

The equation of motion for the (CR3BD) is

$$\ddot{q} = \frac{m_1(x_1 - q)}{|x_1 - q|^3} + \frac{m_2(x_2 - q)}{|x_2 - q|^3} = \frac{\partial}{\partial q} U(q,t),$$

where

$$U(q,t) = \frac{1-\mu}{\sqrt{|q|^2 + \mu^2 - 2\mu|q|\cos(\theta - t)}} + \frac{\mu}{\sqrt{|q|^2 + (1-\mu)^2 + 2(1-\mu)|q|\cos(\theta - t)}}.$$

Revisit: (CR3BD)

Let $$\Delta_1 = \left(\frac{\mu}{|q|}\right)^2 - \frac{2\mu}{|q|}\cos(\theta - t),$$
$$\Delta_2 = \left(\frac{1-\mu}{|q|}\right)^2 + \frac{2(1-\mu)}{|q|}\cos(\theta - t).$$

Then $U(q,t) = \frac{1}{|q|} + U_0(q,t)$ where

$$U_0(q,t) = \frac{(1-\mu)}{|q|}\left[-1 + \frac{\Delta_1}{2} + (1+\Delta_1)^{-\frac{1}{2}}\right] + \frac{\mu}{|q|}\left[-1 + \frac{\Delta_2}{2} + (1+\Delta_2)^{-\frac{1}{2}}\right]$$
$$- \frac{1}{2|q|}\left[(1-\mu)\Delta_1 + \mu\Delta_2\right].$$

Revisit: (CR3BD)

After some calculations, we find

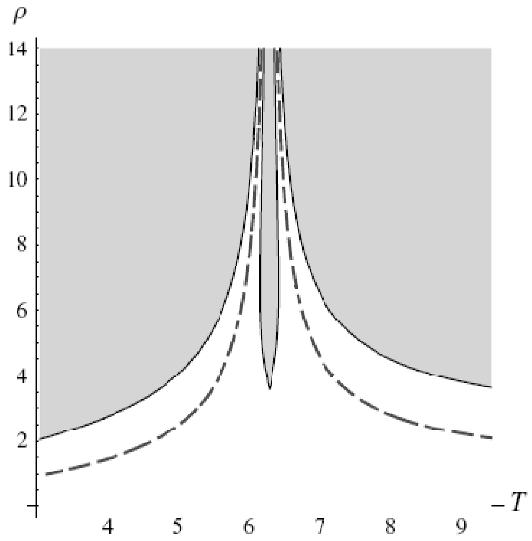
$$R_\rho \leq \frac{1}{\rho^3}\left(\frac{21}{8}+\sqrt{6}\right) \approx \frac{5.0745}{\rho^3} \quad \text{for } \rho \geq 2+\sqrt{6} \approx 4.4495.$$

The (CR4BD) is similar. Given a Lagrangian equilateral relative equilibrium with total mass 1. Consider the motion of a satellite.

$$R_\rho \leq \frac{1}{\rho^3}\left(\frac{8}{3}+\sqrt{6}\right) \approx \frac{5.1162}{\rho^3} \quad \text{for } \rho \geq 2+\sqrt{6} \approx 4.4495.$$

Revisit: (CR3BD)

For (CR3BD), region on which (*) holds:



For (CR4BD), region on which (*) holds looks similar.

Revisit: spacecraft near an asteroid

Back to the model for spacecraft motion about an elliptical asteroid which rotates with unit angular velocity:

$$\ddot{x} - 2\dot{y} = \frac{\partial V}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial V}{\partial y},$$

where $V(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{\sqrt{x^2 + y^2}} + V_{20}(x, y) + V_{22}(x, y).$

In the form (SO), the U_0 is a multiple of (q_1=x,q_2=y)

$$U_0(q, t) = \frac{(q_1^2 - q_2^2)\cos 2t + 2q_1 q_2 \sin 2t}{(q_1^2 + q_2^2)^{5/2}}.$$

Moreover, $U_0(x + iy, 0) = V_{20}(x, y) + V_{22}(x, y).$

Revisit: spacecraft near an asteroid

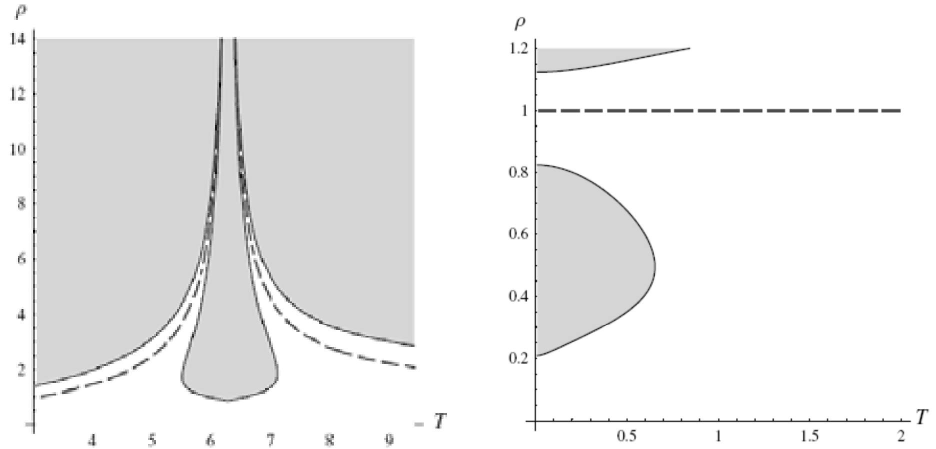
A simple model:

$$V_{20} = 0, \qquad V_{22} = \frac{3C_{22}(x^2 - y^2)}{(x^2 + y^2)^{5/2}}.$$

The ellipticity coefficient C_{22} varies between 0 and 0.05 for physical systems. [Koon-Marsden-Ross-Lo-Scheeres, *2004*]

Then $R_\rho \leq \dfrac{0.15}{\rho^3}.$

Region on which (*),(**) hold:



Ideas of proof

Consider any integer k. In polar form,

$$\mathcal{A}_T(q) = \int_0^T \frac{1}{2}\left[r^2(1+\dot{\theta})^2 + \dot{r}^2\right] + \frac{1}{r} + U_0(r\cos\theta + ir\sin\theta, 0)\,dt$$
$$\leq \int_0^T \frac{1}{2}\left[r^2(1+\dot{\theta})^2 + \dot{r}^2\right] + \frac{1}{r}dt + R_\rho T$$

Consider a special path $r(t) = \left|1 + \frac{2k\pi}{T}\right|^{-\frac{2}{3}}$ and $\theta(t) = \frac{2k\pi t}{T}$
we find an upper bound for the infimum of $\mathcal{A}_T$ is

(UB) $$\frac{3}{2}\left|1 + \frac{2k\pi}{T}\right|^{\frac{2}{3}} T + R_\rho T.$$

Ideas of proof

Assume $\min_{[0,T]} |q(t)| = \underline{r} \geq \rho$, let $\bar{r} = \max_{t\in[0,T]} r(t) = \max_{t\in[0,T]} |q(t)|$.

Then we find a lower bound for $\mathcal{A}_T$:

$$
\begin{aligned}
\mathcal{A}_T(q) &= \frac{1}{2}\int_0^T r^2(1+\dot{\theta})^2\,dt + \frac{1}{2}\int_0^T \dot{r}^2\,dt + \int_0^T \frac{1}{r} + U_0(r\cos\theta + ir\sin\theta, 0)\,dt \\
&\geq \frac{\underline{r}^2}{2}\int_0^T (1+\dot{\theta})^2 dt + \frac{1}{2T}\left(\int_0^T |\dot{r}| dt\right)^2 + \frac{T}{\bar{r}} - R_{\underline{r}}T \\
&\geq \frac{\underline{r}^2}{2T}\left(\int_0^T |1+\dot{\theta}| dt\right)^2 + \frac{1}{2T}(\bar{r} - \underline{r})^2 + \frac{T}{\bar{r}} - R_{\underline{r}}T \\
&\geq \frac{\underline{r}^2}{2T}(T + 2k\pi)^2 + T\kappa(T, \underline{r}) - R_{\underline{r}}T.
\end{aligned}
$$

Then compare this lower bound with (UB) to conclude.

What's good about this proof?

1. This proof relies only on quantitative estimates for distance from the mass center, works for non-point masses.
   In sharp contrast, recent works on (NBD) by variational methods depends on estimates for collision paths.

2. This proof gives upper bound estimates for the distance from the mass center, complementary to classical methods which works only for distant satellites. It also extends the classical Whittaker criterion (1904) on the existence of satellite orbits inside a circular ring.

THE END