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Collinear solution to the general relativistic three-body problem

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Abstract. The three-body problem is reexamined in the framework of general relativity. The Newtonian three-body problem admits *Euler's collinear solution*, where three bodies move around the common center of mass with the same orbital period and always line up. We show that the collinear solution remains true at the first post-Newtonian order with a correction to the spatial separation between masses. Also we prove the uniqueness of the configuration for given system parameters (the masses and the end-to-end length).

1. Euler's collinear solution in the Newton gravity

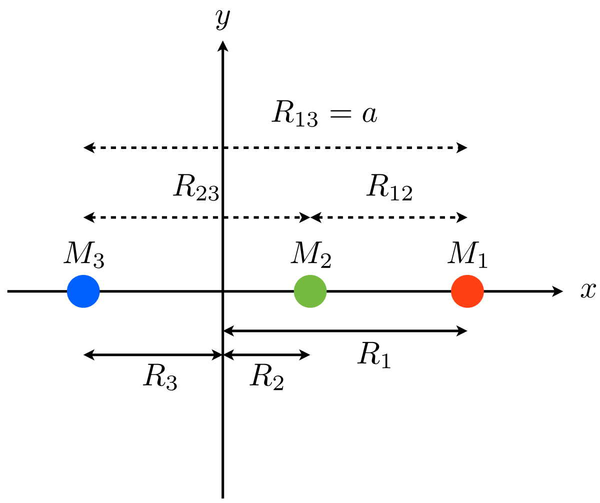

Figure 1. Schematic figure for a classical configuration of three masses denoted by M_1 , M_2 and M_3.

The location of each mass M_I $(I = 1, 2, 3)$ is written as $\boldsymbol{X}_I \equiv (x_I, 0)$. Without loss of generality, we assume $x_3 < x_2 < x_1$. Let R_I define the relative position of each mass with respective to the center of mass $\boldsymbol{X}_G \equiv (x_G, 0)$, namely $R_I \equiv x_I - x_G$ ($R_I \neq |\boldsymbol{X}_I|$ unless $x_G = 0$). We choose $x = 0$ between M_1 and M_3. We thus have $R_3 < R_2 < R_1$, $R_3 < 0$ and $R_1 > 0$.

It is convenient to define an important ratio as $R_{23}/R_{12} = z$. Then we have $R_{13} = (1+z)R_{12}$. The equation of motion in Newton gravity becomes

$$R_1\omega^2 = \frac{M_2}{R_{12}^2} + \frac{M_3}{R_{13}^2}, \tag{1}$$

$$R_2\omega^2 = -\frac{M_1}{R_{12}^2} + \frac{M_3}{R_{23}^2}, \tag{2}$$

$$R_3\omega^2 = -\frac{M_1}{R_{13}^2} - \frac{M_2}{R_{23}^2}, \tag{3}$$

where we define

$$\boldsymbol{R}_{IJ} \equiv \boldsymbol{X}_I - \boldsymbol{X}_J, \quad R_{IJ} \equiv |\boldsymbol{R}_{IJ}|. \tag{4}$$

First, we subtract Eq. (2) from Eq. (1) and Eq. (3) from Eq. (2) and use $R_{12} \equiv |\boldsymbol{X}_1 - \boldsymbol{X}_2|$ and $R_{23} \equiv |\boldsymbol{X}_2 - \boldsymbol{X}_3|$. Next, we compute a ratio between them to delete ω^2. Hence we obtain a fifth-order equation as

$$\begin{aligned}&(M_1 + M_2)z^5 + (3M_1 + 2M_2)z^4 + (3M_1 + M_2)z^3 \\ &- (M_2 + 3M_3)z^2 - (2M_2 + 3M_3)z - (M_2 + M_3) = 0.\end{aligned} \tag{5}$$

Now we have a condition as $z > 0$. Descartes' rule of signs : the number of positive roots either equals to that of sign changes in coefficients of a polynomial or less than it by a multiple of two. According to this rule, Eq. (5) has the only positive root $z > 0$, though such a fifth-order equation cannot be solved in algebraic manners as shown by Galois. After obtaining z, one can substitute it into a difference, for instance between Eqs. (1) and (3). Hence we get ω.

2. What happens in GR ?

2.1. The EIH equation of motion for a many-body system

In order to include the dominant part of general relativistic effects, we take account of the terms at the first post-Newtonian order. Namely, the bodies obey the Einstein-Infeld-Hoffman (EIH) equation of motion as

$$\frac{d^2\boldsymbol{r}_K}{dt^2} = \sum_{A\neq K} \boldsymbol{r}_{AK}\frac{1}{r_{AK}^3}\left[1 - 4\sum_{B\neq K}\frac{1}{r_{BK}} - \sum_{C\neq A}\frac{1}{r_{CA}}\left(1 - \frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)\right.$$
$$\left. + \boldsymbol{v}_K^2 + 2\boldsymbol{v}_A^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_K - \frac{3}{2}\left(\frac{\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK}}{r_{AK}}\right)^2\right]$$
$$- \sum_{A\neq K}(\boldsymbol{v}_A - \boldsymbol{v}_K)\frac{\boldsymbol{r}_{AK}\cdot(3\boldsymbol{v}_A - 4\boldsymbol{v}_K)}{r_{AK}^3}$$
$$+ \frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{1}{r_{AK}r_{CA}^3}.$$

2.2. The seventh-order equation

Similarly to the above Newtonian case, we obtain a seventh-order equation as

$$F(z) \equiv \sum_{k=0}^{7} A_k z^k = 0, \tag{6}$$

where we define the mass ratio as $\nu_I \equiv M_I/M$ for $M \equiv \sum_I M_I$ and

$$A_7 = \frac{M}{a}\left[-4 - 2(\nu_1 - 4\nu_3) + 2(\nu_1^2 + 2\nu_1\nu_3 - 2\nu_3^2) - 2\nu_1\nu_3(\nu_1 + \nu_3)\right],$$

$$A_6 = 1 - \nu_3 + \frac{M}{a}\left[-13 - (10\nu_1 - 17\nu_3) + 2(2\nu_1^2 + 8\nu_1\nu_3 - \nu_3^2)\right.$$
$$\left. + 2(\nu_1^3 - 2\nu_1^2\nu_3 - 3\nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_5 = 2 + \nu_1 - 2\nu_3 + \frac{M}{a}\left[-15 - (18\nu_1 - 5\nu_3) + 4(5\nu_1\nu_3 + 4\nu_3^2)\right.$$
$$\left. + 6(\nu_1^3 - \nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_4 = 1 + 2\nu_1 - \nu_3 + \frac{M}{a}\left[-6 - 2(5\nu_1 + 2\nu_3) - 4(2\nu_1^2 - \nu_1\nu_3 - 4\nu_3^2)\right.$$
$$\left. + 2(3\nu_1^3 + \nu_1^2\nu_3 - 2\nu_1\nu_3^2 - 3\nu_3^3)\right],$$

$$A_3 = -(1 - \nu_1 + 2\nu_3) + \frac{M}{a}\left[6 + 2(2\nu_1 + 5\nu_3) + 4(-4\nu_1^2 - \nu_1\nu_3 + 2\nu_3^2) - 2(-3\nu_1^3 - 2\nu_1^2\nu_3 + \nu_1\nu_3^2 + 3\nu_3^3)\right],$$

$$A_2 = -(2 - 2\nu_1 + \nu_3) + \frac{M}{a}\left[15 + (-5\nu_1 + 18\nu_3) - 4(4\nu_1^2 + 5\nu_1\nu_3) - 6(-\nu_1^3 - \nu_1^2\nu_3 + \nu_3^3)\right],$$

$$A_1 = -(1 - \nu_1) + \frac{M}{a}\left[13 + (-17\nu_1 + 10\nu_3) - 2(-\nu_1^2 + 8\nu_1\nu_3 + 2\nu_3^2) - 2(-\nu_1^3 - 3\nu_1^2\nu_3 - 2\nu_1\nu_3^2 + \nu_3^3)\right],$$

$$A_0 = \frac{M}{a}\left[4 + 2(-4\nu_1 + \nu_3) - 2(-2\nu_1^2 + 2\nu_1\nu_3 + \nu_3^2) + 2\nu_1\nu_3(\nu_1 + \nu_3)\right].$$

This seventh-order equation is symmetric for exchanges between ν_1 and ν_3, only if one makes a change $z \to 1/z$. This symmetry seems to validate the complicated form of each coefficient.

3. Uniqueness of solutions

The seventh-order equation has at most three positive roots, which apparently provide three cases of the distance ratio. Here we show that the remaining two of the three positive roots must be discarded. Let the smaller root and the larger one be denoted as z_S and z_L, respectively.

First, we consider the smaller positive root z_S, where we assume $z_S \ll 1$. Then, the seventh-order equation is approximated as

$$A_1 z_S + A_0 = 0.$$

We thus obtain an approximate form of the smaller root z_s that leads to

$$\omega_S = O\left(\frac{1}{a}\right),$$

though $\omega_N^2 = O(M/a^3)$ for the Newtonian case. This ω_S implies an extremely fast rotation, since the rotational velocity becomes as

$$v_S \approx a\omega_S = O(1),$$

namely, comparable to the speed of light. This unacceptable branch of such an extremely fast motion contradicts with the post-Newtonian approximation. Hence, z_S must be abandoned.

Similarly, the larger positive root z_L also must be discarded.

As a result, two of the three positive roots are discarded as unphysical ones. Hence, we complete the proof of the uniqueness.

4. Conclusion

We obtained a general relativistic version of Euler's collinear solution for the three-body problem at the post-Newtonian order. Studying global properties of the seventh-order equation that we have derived is left as future work.

It is interesting also to include higher post-Newtonian corrections, especially 2.5PN effects in order to elucidate the secular evolution of the orbit due to the gravitational radiation reaction at the 2.5PN order. One might see probably a shrinking collinear orbit as a consequence of a decrease in the total energy and angular momentum, if such a radiation reaction effect is included. This is a testable prediction.

It may be important also to search other solutions, notably a relativistic counterpart of the Lagrange's triangle solution (so-called L_4 and L_5 in the restricted three-body problem). Clearly it seems much more complicated to obtain relativistic corrections to the Lagrange orbit.

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