Morse index of periodic solutions in the n-body problem

Mitsuru SHIBAYAMA (Mathematical Sciences, Osaka University)

E-mail: shibayama@sigmath.es.osaka-u.ac.jp

URL: http://www.sigmath.es.osaka-u.ac.jp/~shibayama

1 Variational formulation

We consider the classical n-body problem for which the equation of motion is given by

$$m_\ell \ddot{q}_\ell = \frac{\partial V}{\partial q_\ell}, \quad q_\ell \in \mathbb{R}^3, \quad \ell = 1, 2, \ldots, n$$

where an overdot denotes differentiation with respect to the time variable, m_ℓ (> 0) is the ℓ-th mass and

$$V(q_1, \ldots, q_n) = \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|}$$

represents the (negative-)potential energy with the unit gravitational constant.

The n-body problem is equivalent to the variational problem with respect to the action functional

$$\mathcal{A}(q) = \int_0^T L(q, \dot{q}) dt$$

where the function L is the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2} \sum m_k \|\dot{q}_k\|^2 + \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|}.$$

Denote the configuration space by $\hat{\mathcal{X}}$ where

$$\mathcal{X} = \{q = (q_1, \ldots, q_n) \in (\mathbb{R}^3)^n \mid \sum_{k=1}^n m_k q_k = 0\}$$

$$\Delta_{ij} = \{q \in \mathcal{X} \mid q_i = q_j\}, \quad \Delta = \bigcup_{i<j} \Delta_{ij}, \quad \hat{\mathcal{X}} = \mathcal{X} - \Delta$$

and let

$$\Lambda = H^1(\mathbb{R}/T\mathbb{Z}, \hat{\mathcal{X}}).$$

2 Symmetric constraint

Assume the all masses are equal.

Let G be a group and let

$$\tau : G \to O(2), \quad \rho : G \to O(3), \quad \sigma : G \to \mathfrak{S}_n,$$

be homomorphisms. We define the action of G to Λ by

$$g \cdot ((q_1, \ldots, q_n)(t)) = (\rho(g) q_{\sigma(g^{-1})(1)}, \ldots, \rho(g) q_{\sigma(g^{-1})(n)})(\tau(g^{-1})t)$$

for $g \in G$ and $q(t) = (q_1, \ldots, q_n)(t) \in \Lambda$. Let

$$\Lambda^G = \{q \in \Lambda \mid g \cdot q = q\}, \quad \mathcal{A}^G = \mathcal{A}|_{\Lambda^G}.$$

3 Choreographic constraint

We take G as the cyclic group $C_n = \langle g \mid g^n = 1 \rangle$ of order n. The homomorphisms are defined by

$$\tau(g) = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \qquad \rho(g) = E_3 \qquad \sigma(g) = (1, 2, \ldots, n).$$

Theorem (Barutello-Terracini, Nonlinearity 04)

The minimizers of $\mathcal{A}^{C_n}$ are just rotating regular n-gons.

4 Symmetric choreographic constraint

We take G as the dihedral group $D_n = \langle g_1, g_2 \mid g_1^2 = g_2^n = (g_1 g_2)^2 = 1\rangle$. The homeomorphisms ρ, σ, τ are defined by

$$\tau(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(g_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\sigma(g_1) = (2, n)(3, n-1)\ldots([\frac{n-1}{2}] + 1, n - [\frac{n-1}{2}] + 1)$$

$$\tau(g_2) = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}, \qquad \rho(g_2) = E_3, \qquad \sigma(g_2) = (1, 2, \ldots, n).$$

Because $\Lambda^{D_n} \subset \Lambda^{C_n}$ and Λ^{D_n} includes the rotating regular n-gons, the minimizers of $\mathcal{A}^{D_n}$ are the rotating regular n-gons.

The set of minimizers is $R_+ \sqcup R_-$ where R_+ (resp. R_-) is the set of the solutions with positive (resp. negative) z-component of $\dot{q}_1(0)$. R_+ and R_- are topologically equivalent to S^1.

5 Mountain Pass Solution

Let

$$\Gamma = \{\gamma \in C([0,1], \Lambda^{D_n}) \mid \gamma(0) \in R_-, \gamma(1) \in R_+\}$$

and let

$$d = \inf_{\gamma \in \Gamma} \max_{q \in \gamma([0,1])} \mathcal{A}(q).$$

Theorem (S.) **There is a periodic solution which attains d. The solution has at most one collision.**

Outline of the proof

Applying the mountain pass theorem on the set of curves connecting these two components, there is a mountain pass solution. The Morse index of the mountain pass solution is no more than 1.

On the other hand for any solution with a collision, we can make the modified curves with lower value of action functional in two orthogonal direction in Λ^{D_n}. Thus the mountain pass solution has no collision.

The other trivial solutions are the rotating regular n-gon which rotates several times per the period T. Morse index of these solutions are greater than 1 if $n = 2m$. $\square$

6 Minimax solution

Let

$$\Omega = \{f : D \to \Lambda^{D_n} \mid f|_{\partial D} \in \mathrm{Homeo}(\partial D, R_+)\}$$

and let

$$c = \inf_{f \in \Omega} \max_{q \in f(D)} \mathcal{A}(q).$$

Theorem (S.)

There is a periodic solution which attains c. The solution may have at most two collisions.

Outline of the proof

The existence follows from the minimax theorem. The Morse index is no more than 2.

Morse index of the other trivial solutions are greater than 2 if $n = 6m$. $\square$