Level Repulsion and threefold degeneracy of eigenstates of the Liouvillian in the Kirkwood gaps in Asteroid belt

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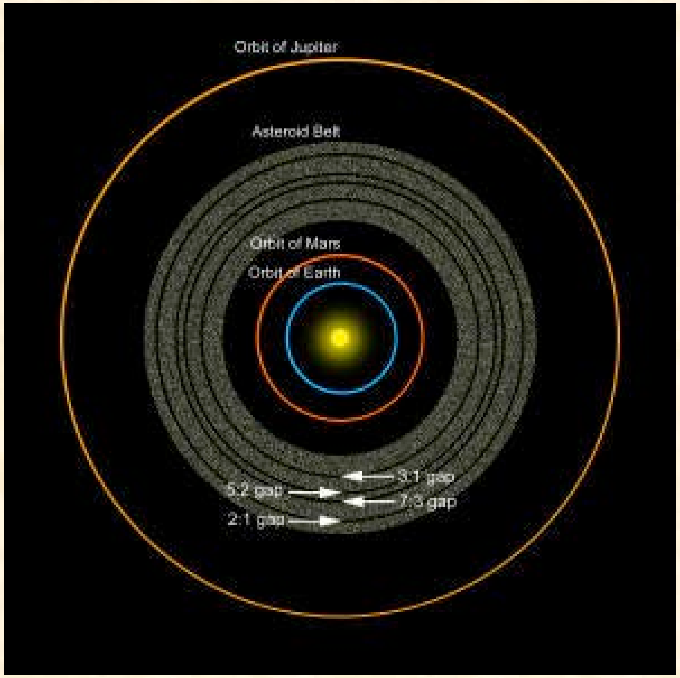
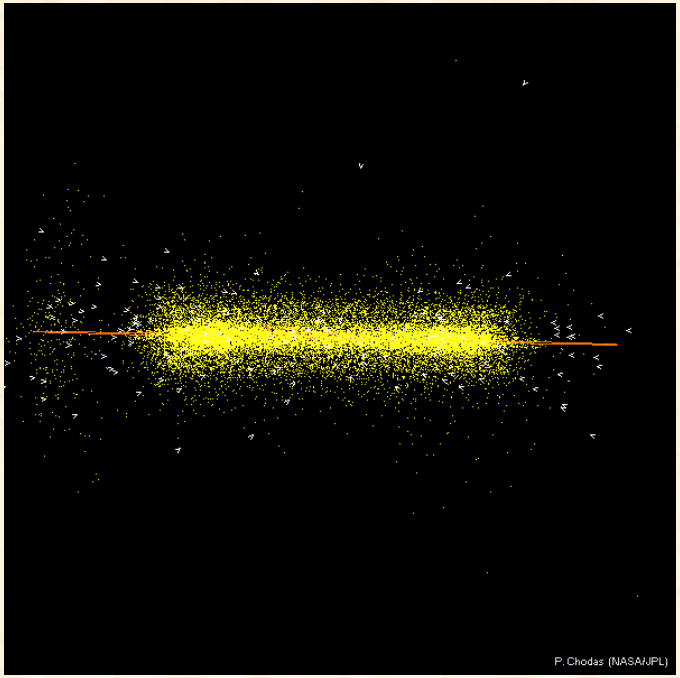
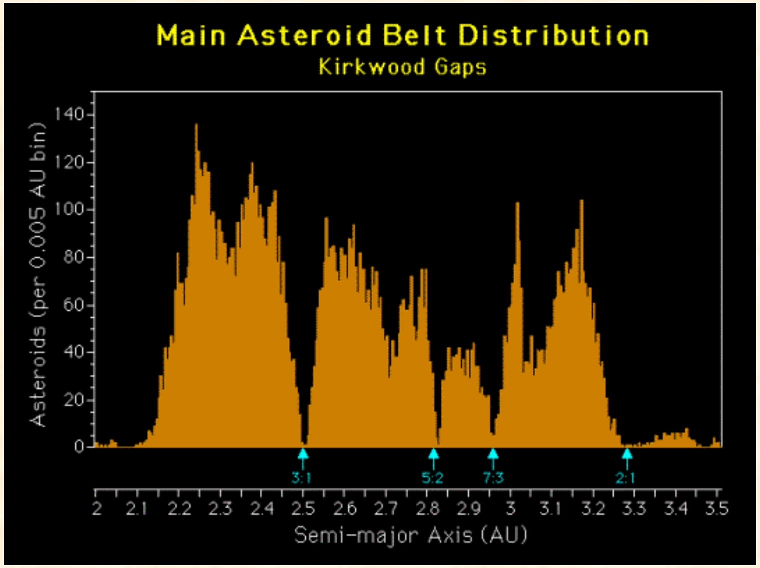
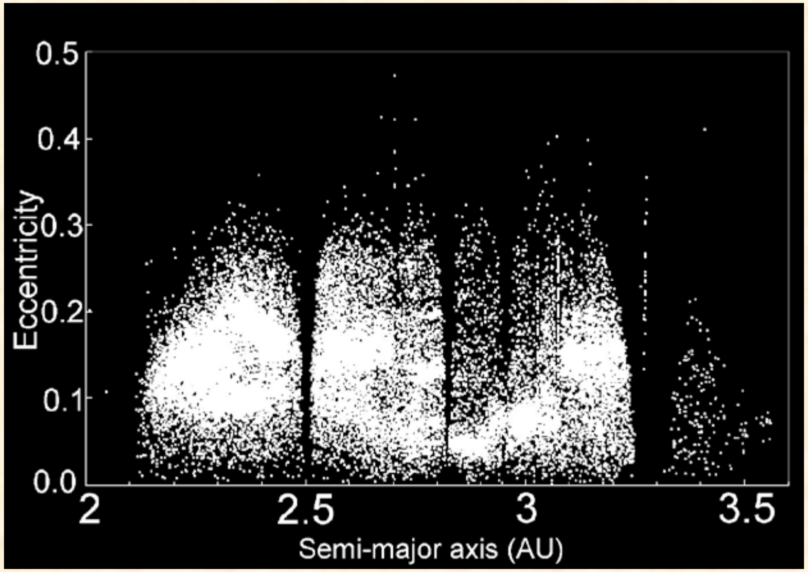
Asteroid belt





The Kirkwood gaps





D. Kirkwood, *Meteoric Astronomy: A Tdreatise on Shooting-Stars, Fireballs, And Aerolites* (J. B. Lippincott, Piladelphia, 1867)

The Liouville equation

Hamiltonian: $$H = H_0(\vec{p}) + \lambda V(\vec{x})$$

$$\vec{x} \equiv (\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N)$$

The Liouville equation

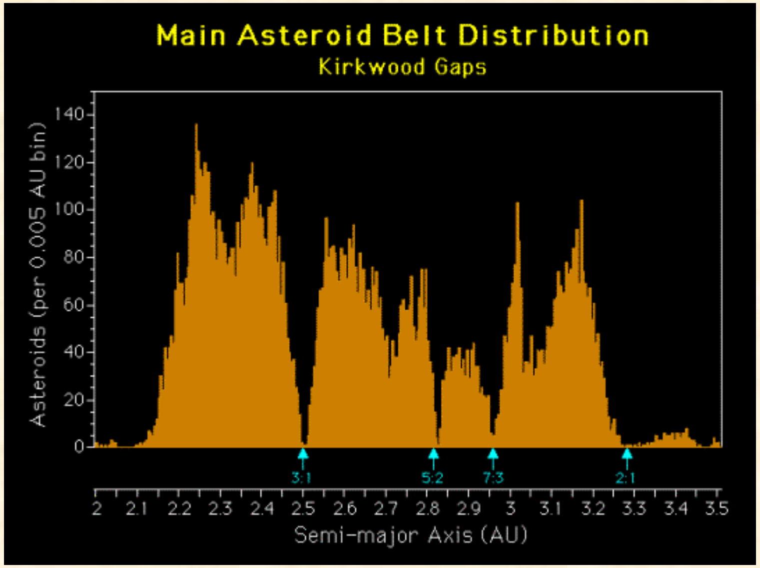
$$i\frac{\partial \rho}{\partial t} = L_H \rho$$

Liouvillian: a Poisson bracket with the Hamiltonian

$$L_H \rho = i\{H, \rho\} \equiv i\sum_j^N \left(\frac{\partial H}{\partial \mathbf{x}_j}\frac{\partial}{\partial \mathbf{p}_j} - \frac{\partial H}{\partial \mathbf{p}_j}\frac{\partial}{\partial \mathbf{x}_j} \right) \rho(\vec{p}, \vec{x})$$

Physical dimension: frequency

$$L_H = L_0 + \lambda L_V$$



Band spectrum of the Liouvillian
in the restrict problem of three bodys

Keplar's third law

$$\omega \propto a^{-3/2}$$

a : semi - major axis

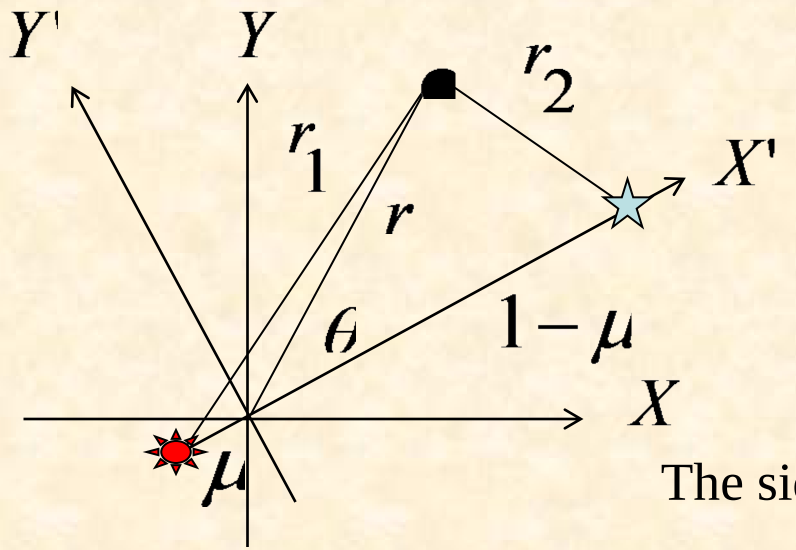
Hamiltonian of the restricted three-body problem

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - p_\theta - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

$$r_1 = \sqrt{r^2 - 2\mu r\cos\theta + \mu^2}$$

$$r_2 = \sqrt{r^2 - 2(1-\mu) r\cos\theta + (1-\mu)^2}$$

The synodic coordinate



$$\mu \equiv \frac{M_J}{M_S + M_J} \approx 0.95 \times 10^{-3}$$

The sidereal coordinate

Delaunay's variables: (L,l), (G,g)

$$H = -\frac{1}{2L^2} - G + \mu V(L,G,l,g;\mu)$$

$$\mu V(L,G,l,g;\mu) = \frac{1}{r} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

$L = \sqrt{a}$

l : mean anomaly

G : angler momentum of the asteroid

g : argument of the perihelion in the rotating system

$$G = p_\theta = \sqrt{a(1-\varepsilon^2)}$$

$$\varepsilon = \sqrt{1-\left(\frac{G}{L}\right)^2}$$: eccentricity

$$l = u - \varepsilon \sin u \quad : \text{Keplar's Equation}$$

$$u : \text{eccentric anomaly}$$

$$r = a(1 - \varepsilon \cos u) = \frac{a(1-\varepsilon^2)}{1+\varepsilon \cos f}$$

$$\cos u = \frac{\varepsilon + \cos f}{1 + \varepsilon \cos f}$$

$$f = \theta - g : \text{true anomaly}$$

$$\cos f = -\varepsilon + \frac{2(1-\varepsilon^2)}{\varepsilon} \sum_{n=1}^{\infty} J_n'(n\varepsilon) \cos(nl)$$

$$\sin f = 2\sqrt{1-\varepsilon^2} \sum_{n=1}^{\infty} J_n'(n\varepsilon) \sin(nl)$$

$$\frac{1}{r_1} = \frac{1}{r} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{\mu}{r}\right)^n \quad \text{for} \quad \mu << 1,\ r \sim 1$$

$$\frac{1}{r_2} = \frac{1}{1-\mu} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r}{1-\mu}\right)^n \quad \text{for} \quad \mu << 1,\ 0 < r < 1$$

$$H = H_0 + \mu V \Rightarrow L_H = L_0 + \mu L_V$$

$$L_0 = -i\omega_l \frac{\partial}{\partial l} - i\omega_g \frac{\partial}{\partial g}$$

$$\omega_l = \frac{\partial H_0}{\partial L} = \frac{1}{L^3}$$

$$\omega_g = \frac{\partial H_0}{\partial G} = -1$$

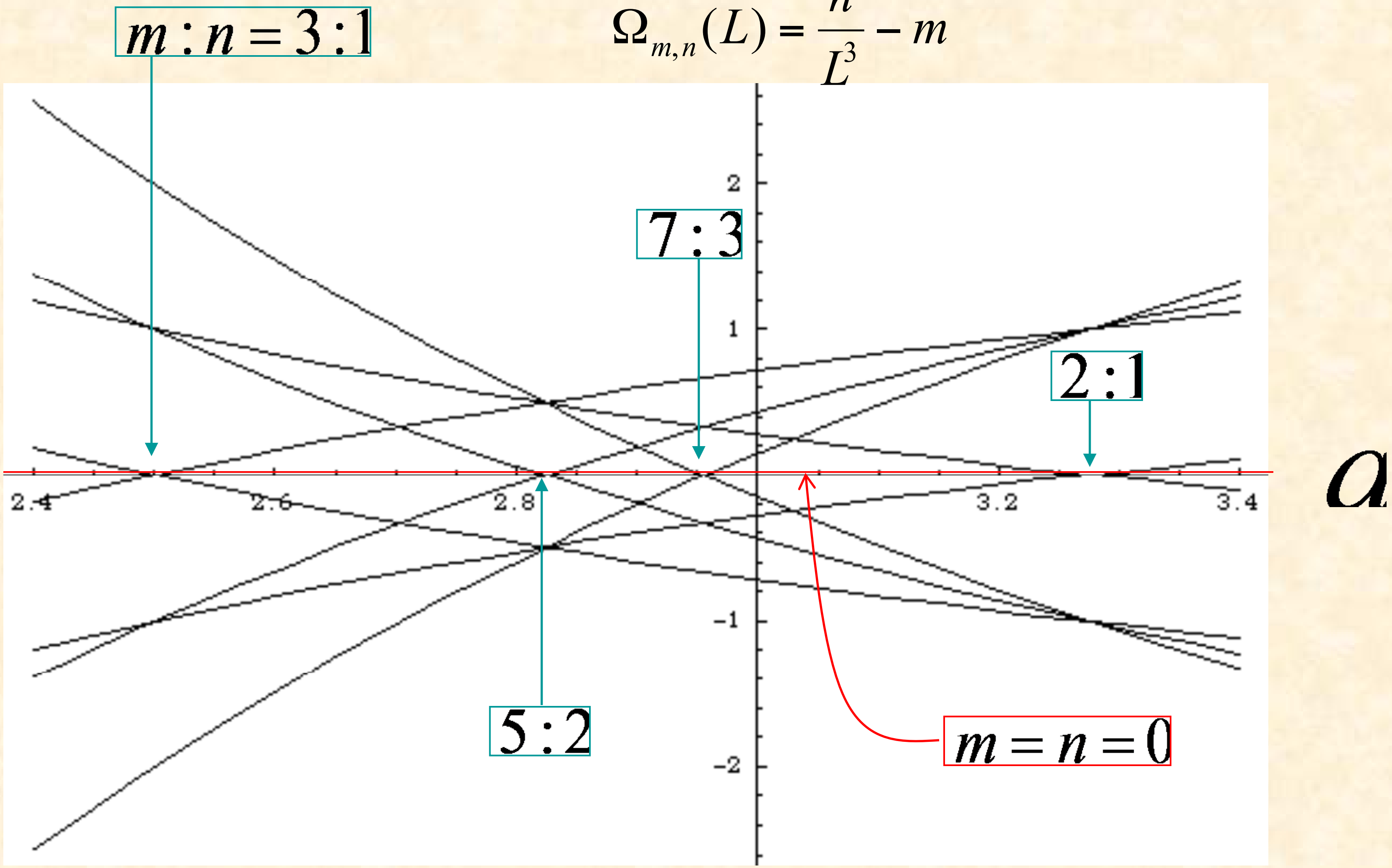
Eigenvalue states for the two-body problem

$$L_0 \mathrm{e}^{i(nl+mg)} = \left(\frac{n}{L^3} - m\right)\mathrm{e}^{i(nl+mg)} \qquad m, n = 0, \pm 1, \pm 2, \cdots$$

Dispersion relation for *m-n* mode: $\Omega = \Omega_{m,n}(L)$

$$L = \sqrt{\frac{a}{5.2\text{AU}}}$$

$$\Omega_{m,n}(L) = \frac{n}{L^3} - m$$



Threefold degeneracy

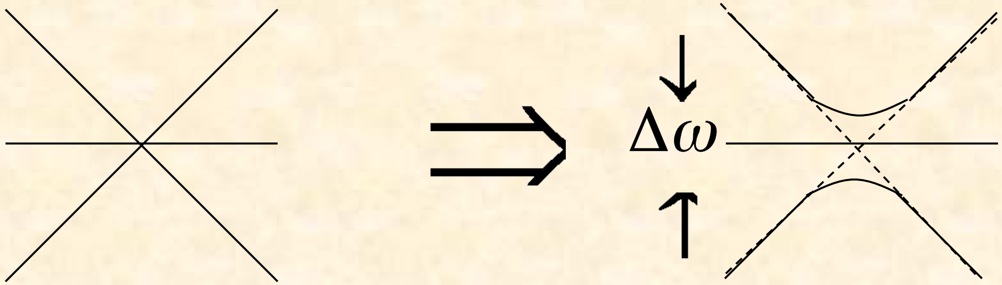
The degenerate perturbation theory for resonances

$$
\begin{array}{c} \\ -3:-1 \\ 0:0 \\ 3:1 \end{array}
\begin{array}{c} \begin{array}{ccc} -3:-1 & 0:0 & 3:1 \end{array} \\
\begin{pmatrix} 0 & \mathcal{L}_{-3-1} & 0 \\ \mathcal{L}^{*}_{-3\text{-}1} & 0 & \mathcal{L}_{31} \\ 0 & \mathcal{L}^{*}_{31} & 0 \end{pmatrix} \end{array}
\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}
\qquad \lambda = 0, \quad \pm\sqrt{\left|\mathcal{L}_{-3-1}\right|^2 + \left|\mathcal{L}_{31}\right|^2}
$$

Level repulsion

Mode coupling
(Selection rule)

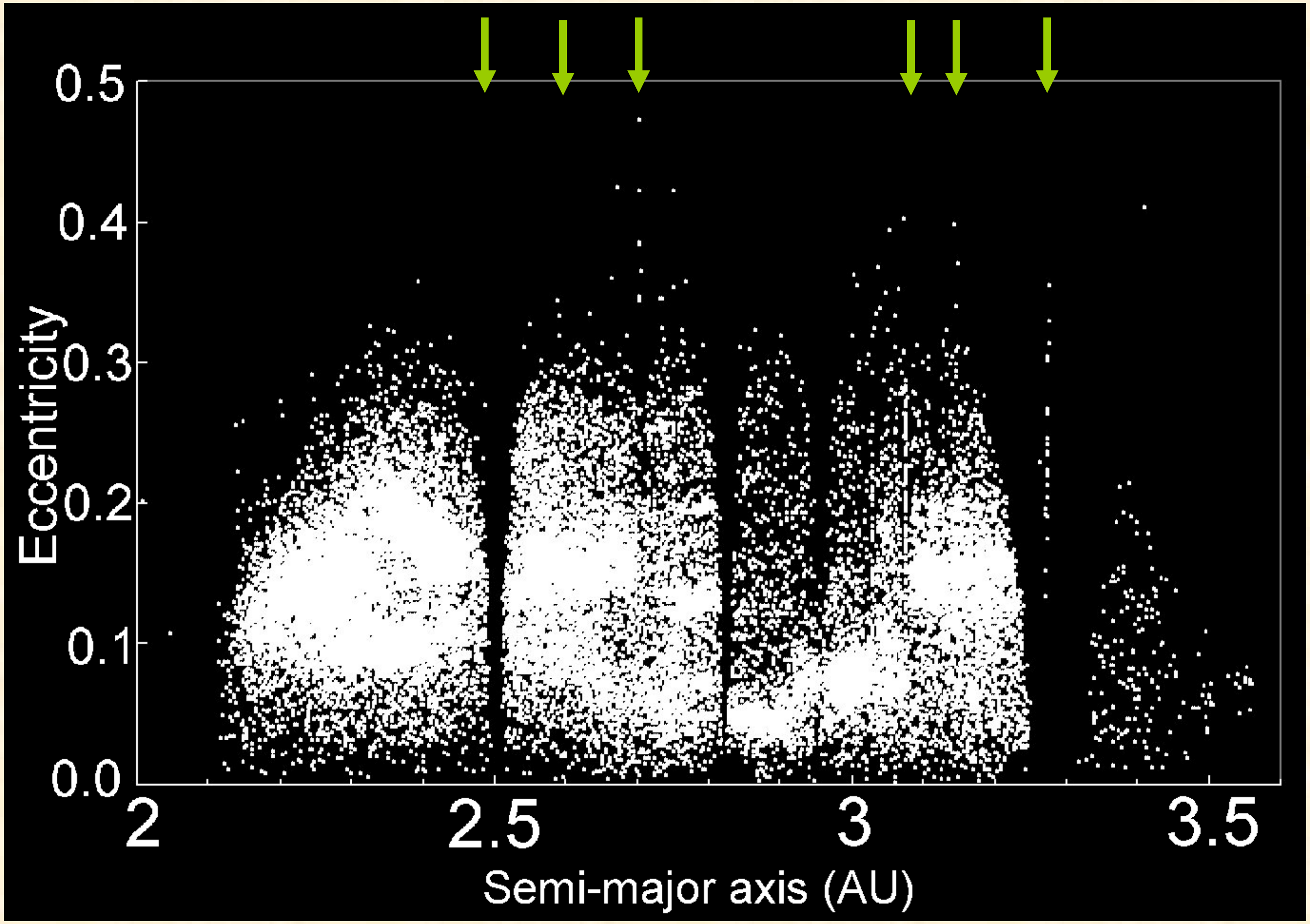
Saturn’s ring?



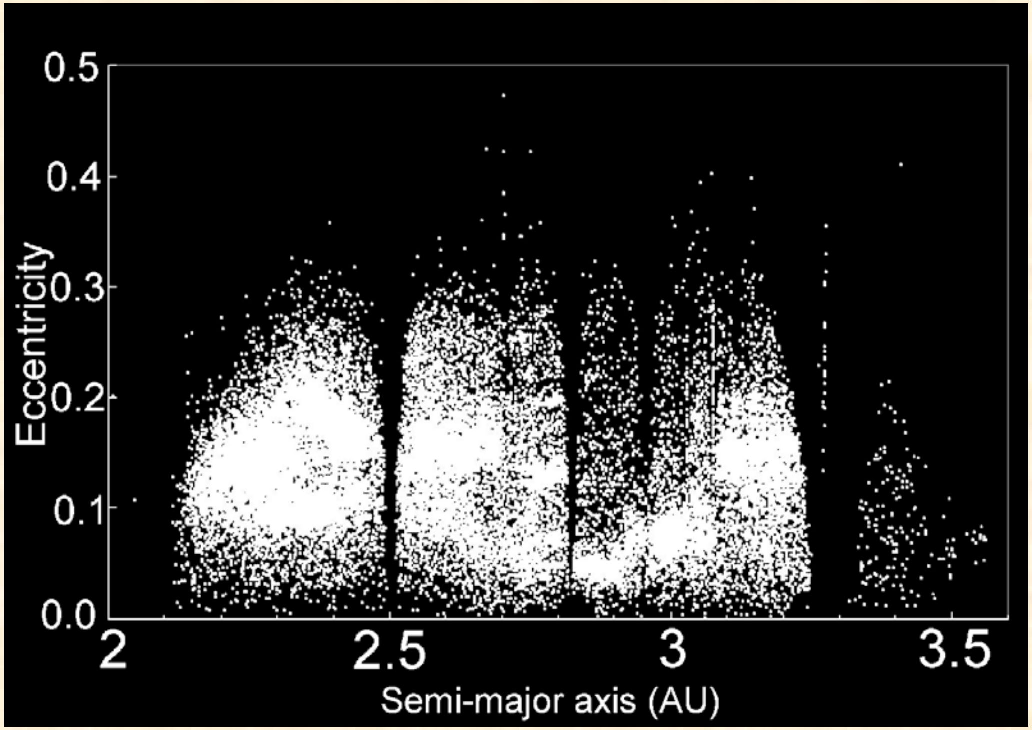
$$
2:1 \qquad 3:1 \qquad 5:2 \qquad 7:3 \qquad \cdots \qquad m:n
$$

$$
\mathcal{L}_{31} \sim \quad \mu a^{3/2}\varepsilon^{-1} \qquad \mu a^{5/2}\varepsilon^{0} \qquad \mu a^{9/2}\varepsilon^{1} \qquad \mu a^{13/2}\varepsilon^{2} \qquad \cdots \qquad \mu a^{m-\frac{1}{2}}\varepsilon^{m-n-2}
$$

!



Evidence of the threefold degeneracy



$$|\Delta a_{21}| \sim 0.6 \times 10^{-2} \text{ AU}$$

$$|\Delta a_{31}| \sim 1.8 \times 10^{-3} \text{ AU}$$

$$|\Delta a_{12}| \sim 1.5 \times 10^{-4} \text{ AU}$$

Estimation of the lower bound!

The effect of the eccentricity of Jupiter

eccentricity ~ 0.05

The elliptic restricted problem

The Floquet quasi-eigenvalue problem