Conservative Discretization of the Gravitational Three-Body Problem

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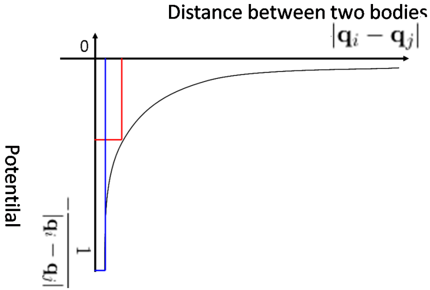
Features of General Three Body Problem

- **Singularities** : A Source of large error



Introduction

Features of General Three Body Problem

- **Conserved quantities** : Hamiltonian, linear momentum, the position of center of mass, angular momentum

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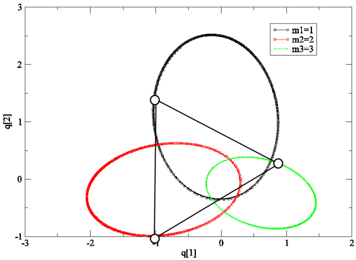
Features of General Three Body Problem

- **Periodic orbits** : orbits corresponding to Lagrangian triangle solutions, figure-eight choreography, Broucke's periodic orbits .....



Orbits corresponding to Lagrangian triangle solutions

Introduction

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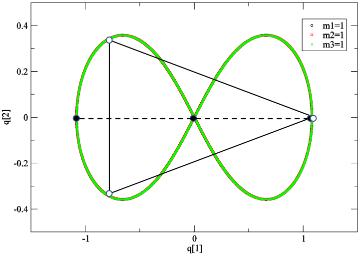
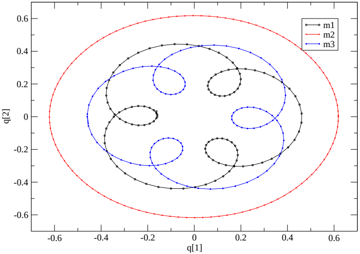


Figure-eight choreography

Features of General Three Body Problem

- **Periodic orbits** : orbits corresponding to Lagrangian triangle solutions, figure-eight choreography, Broucke's periodic orbits .....



Broucke's periodic orbits

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Comparison between known integrators

	Eliminating singularities	Conservation	Re-establiehsed particular solutions
Energy-momentum integrator	**No**	Complete	Circular Lagrangian, eight (P.7 Graph)
Symplectic integrator	No	Imcomplete	Circular Lagrangian, eight
NBODY $1-6$	Yes	Imcomplete	Almost of all Lagrangian, eight

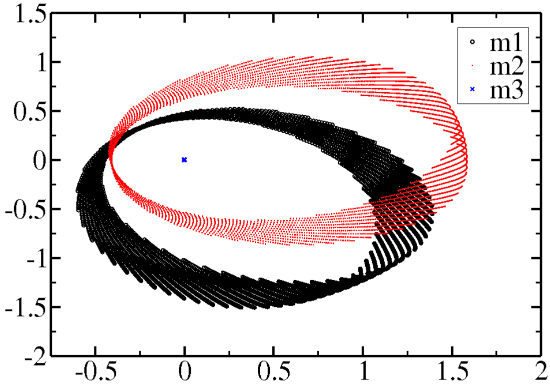
Energy-momentum integrator: T. Matsuo, D. Furihata, D. Greenspan, O.Gonzalez, ...

Symplectic integrator: H. Yoshida, R.D. Ruth, F. Kang,...

NBODY1-6: S. Aarseth, K. Zare, K. Nitadori

Introduction

Destruction of orbits due to numerical integration



Orbits of an elliptic Lagrangian triangle solution
by an energy-momentum integrator [Greenspan's method (1974)]

Infuluence of singularities !

Introduction

Features of General Three Body Problem

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Comparison between known integrators and proposed method (d-GTBP)

	Eliminating singularities	Conservation	Re-establieshed particular solutions
Energy-momentum integrator	**No**	Complete	Circular Lagrangian, eight
Symplectic integrator	No	Imcomplete	Circular Lagrangian, eight
NBODY $1-6$	Yes	Imcomplete	Almost of all Lagrangian, eight
Proposed method (d-GTBP)	**Yes**	**Complete**	**All solutions described above**

GTBP in the barycentric inertial frame

Barycentric inertial frame

- **Canonical eq.**

$$\frac{d}{dt}\mathbf{q}_i = \frac{\mathbf{p}_i}{m_i}$$

$$\frac{d}{dt}\mathbf{p}_i = -m_j \frac{\mathbf{q}_i - \mathbf{q}_j}{|\boxed{\mathbf{q}_i} - \boxed{\mathbf{q}_j}|^3} \cdots$$

In the case of two-body collisions, we need to **consider two vectors simultaneously.**

$$-m_k \frac{\mathbf{q}_i - \mathbf{q}_k}{|\mathbf{q}_i - \mathbf{q}_k|^3}, \quad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$$

- **Hamiltonian**

$$H = \sum_{(i,j,k)} \left[\frac{|\mathbf{p}_i^2|}{2m_i} - \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|} \right]$$

GTBP in the barycentric inertial frame

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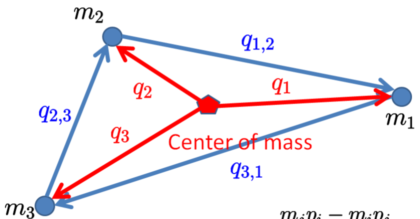
$$\frac{d}{dt}\mathbf{p}_i = -m_j\frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} - m_k\frac{\mathbf{q}_i - \mathbf{q}_k}{|\mathbf{q}_i - \mathbf{q}_k|^3}$$

$$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$$

- **Hamiltonian**

$$H = \sum_{(i,j,k)}\left[\frac{|\mathbf{p}_i^2|}{2m_i} - \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}\right]$$



$$q_{i,j} = q_i - q_j, \quad p_{i,j} = \frac{m_j p_i - m_i p_j}{M},$$
$$M = m_1 + m_2 + m_3$$

Fig. The barycentric and relative coordinate system

GTBP in a relative coordinate system

Relative coordinate system

- **Constrained canonical eq.**

[Constrained formulation with Lagrangian multipliers]

$$\frac{d}{dt}\mathrm{q}_{i,j} = \frac{M}{m_i m_j}\mathrm{p}_{i,j}, \qquad \frac{d}{dt}\mathrm{p}_{i,j} = -\frac{m_i m_j}{|\mathrm{q}_{i,j}|^3}\mathrm{q}_{i,j} - \left(\frac{\partial \phi(\mathrm{q})}{\partial \mathrm{q}}\right)^\top \boldsymbol{\lambda}$$

$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2),$ $\boldsymbol{\lambda}$: Lagrangian multipliers

- **Constraint**

$\phi(\mathrm{q}) \equiv \mathrm{q}_{1,2} + \mathrm{q}_{2,3} + \mathrm{q}_{3,1} = 0$

- **Hamiltonian**

$$H = \sum_{(i,j,k)} \left[\frac{M}{2m_i m_j}|\mathrm{p}_{i,j}|^2 - \frac{m_i m_j}{|\mathrm{q}_{i,j}|}\right] + \phi(\mathrm{q})^\top \boldsymbol{\lambda}$$

GTBP in a relative coordinate system

Relative coordinate system

- **Constrained canonical eq.**

[Constrained formulation with Lagrangian multipliers]

$$\frac{d}{dt}\mathrm{q}_{i,j} = \frac{M}{m_i m_j}\mathrm{p}_{i,j}, \qquad \frac{d}{dt}\mathrm{p}_{i,j} = -\frac{m_i m_j}{|\mathrm{q}_{i,j}|^3}\mathrm{q}_{i,j} - I_2\boldsymbol{\lambda}$$

$$(i,j,k) = (1,2,3), (2,3,1), (3,1,2)$$

In the case of two-body collisions, we **have only to consider** just one vector.

- **Constraint**

$$\phi(\mathrm{q}) \equiv \mathrm{q}_{1,2} + \mathrm{q}_{2,3} + \mathrm{q}_{3,1} = 0$$

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$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$

- **Constraint**

$$\phi(\mathbf{q}) \equiv \mathbf{q}_{1,2} + \mathbf{q}_{2,3} + \mathbf{q}_{3,1} = 0$$

$$\Longrightarrow \frac{d^2}{dt^2}\phi(\mathbf{q}) = \frac{M}{m_1 m_2 m_3}\left(m_3\frac{d}{dt}\mathbf{p}_{1,2} + m_1\frac{d}{dt}\mathbf{p}_{2,3} + m_2\frac{d}{dt}\mathbf{p}_{3,1}\right) = 0$$

- **Hamiltonian**

$$H = \sum_{(i,j,k)}\left[\frac{M}{2m_i m_j}|\mathbf{p}_{i,j}|^2 - \frac{m_i m_j}{|\mathbf{q}_{i,j}|}\right] + \phi(\mathbf{q})^\top\boldsymbol{\lambda}$$

GTBP in a relative coordinate system

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- **Constraint**

$$\phi(\mathrm{q}) \equiv \mathrm{q}_{1,2} + \mathrm{q}_{2,3} + \mathrm{q}_{3,1} = 0$$

$$\Longrightarrow \boldsymbol{\lambda} = -\frac{m_1 m_2 m_3}{M} \sum_{(i,j,k)} \frac{\mathrm{q}_{i,j}}{|\mathrm{q}_{i,j}|^3}$$

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$$(i,j,k) = (1,2,3), (2,3,1), (3,1,2)$$

↑ Singularity Point : A Source of large error

- **Constraint**

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$$\Longrightarrow \lambda = -\frac{m_1 m_2 m_3}{M} \sum_{(i,j,k)} \frac{\mathrm{q}_{i,j}}{|\mathrm{q}_{i,j}|^3}$$

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Heggie's Regularization of GTBP

Before regularization

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LC tr. for every pair (i, j)
(D. C. Heggie, 1974)

$$q_{i,j[1]} = Q_{i,j[1]} Q_{i,j[2]},$$

$$q_{i,j[2]} = \frac{1}{2} \left(Q^2_{i,j[1]} - Q^2_{i,j[2]} \right)$$

$$p_{i,j[1]} = \frac{P_{i,j[1]} Q_{i,j[2]} + P_{i,j[2]} Q_{i,j[1]}}{|\mathbf{Q}_{i,j}|^2}$$

$$p_{i,j[2]} = \frac{P_{i,j[1]} Q_{i,j[1]} - P_{i,j[2]} Q_{i,j[2]}}{|\mathbf{Q}_{i,j}|^2}$$

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$\Downarrow$ LC. tr

After Regularization

- **Hamiltonian**: $H = \sum_{(i,j,k)} \frac{1}{|\mathbf{Q}_{i,j}|^2} \left[\frac{M}{2m_i m_j} |\mathbf{P}_{i,j}|^2 - 2m_i m_j \right] + \Phi(\mathbf{Q})^\top \lambda$
- **Constraint**: $\Phi(\mathbf{Q}) = \left[\sum_{(i,j,k)} Q_{i,j[1]} Q_{i,j[2]}, \frac{1}{2} \sum_{(i,j,k)} \left\{ Q_{i,j[1]}^2 - Q_{i,j[2]}^2 \right\} \right]^\top = 0$

Energy-momentum Integration

Continuous-time Hamiltonian system

$H(\mathrm{p},\mathrm{q}) = \text{const.}$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ **Hamiltonian is conserved.**

$\Updownarrow$

$$\frac{d}{dt}H = \sum_{i=1}^{n}\left(\frac{dp_i}{dt}\frac{\partial H}{\partial p_i} + \frac{dq_i}{dt}\frac{\partial H}{\partial q_i}\right) \equiv 0 \quad \Longleftarrow$$

Autonoumous Hamiltonian eq. :
$$\begin{cases} \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i}, \\ \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i} \end{cases}$$

$\Downarrow$ Discretization

Discrete-time Hamiltonian system D. Greenspan, T. Matuso, D. Furihata, R. I. McLachlan, ...

$\delta H \equiv H(\mathrm{p}^{(k+1)},\mathrm{q}^{(k+1)}) - H(\mathrm{p}^{(k)},\mathrm{q}^{(k)}) = 0$ $\cdots\cdots$ **Hamiltonian is conserved.**

$\Updownarrow$

$$\frac{H^{(k+1)} - H^{(k)}}{t^{(k+1)} - t^{(k)}} = \sum_{i=1}^{n}\left(\frac{p_i^{(k+1)} - p_i^{(k)}}{t^{(k+1)} - t^{(k)}}\boxed{\frac{\delta H}{\delta p_i}} + \frac{q_i^{(k+1)} - q_i^{(k)}}{t^{(k+1)} - t^{(k)}}\boxed{\frac{\delta H}{\delta q_i}}\right) \equiv 0$$

discrete partial derivative

$\Uparrow$

Discrete-time Hamiltonian eq. :
$$\frac{p_i^{(k+1)} - p_i^{(k)}}{t^{(k+1)} - t^{(k)}} = -\boxed{\frac{\delta H}{\delta q_i}}, \quad \frac{q_i^{(k+1)} - q_i^{(k)}}{t^{(k+1)} - t^{(k)}} = \boxed{\frac{\delta H}{\delta p_i}}$$

Regularized GTBP and BEC

Regularized GTBP

$$\frac{d}{dt}\mathbf{Q}_{i,j} = \frac{M}{m_i m_j}\frac{\mathbf{P}_{i,j}}{|\mathbf{Q}_{i,j}|^2}, \quad \Phi(\mathbf{Q}) = 0$$

$$\frac{d}{dt}\mathbf{P}_{i,j} = \frac{\mathbf{Q}_{i,j}}{|\mathbf{Q}_{i,j}|^4}\left(\frac{M}{2m_i m_j}|\mathbf{P}_{i,j}|^2 - 2m_i m_j\right) - \begin{bmatrix} Q_{i,j[2]} & Q_{i,j[1]} \\ Q_{i,j[1]} & -Q_{i,j[2]} \end{bmatrix}\lambda$$

⇓ **Energy-momentum integrator**

Basic Energy Conservative scheme (Gonzalez, 1999 ∈ Matsuo, Furihata)

order 2, time revseribile

$$\frac{\mathbf{Q}_{i,j}^{(k+1)} - \mathbf{Q}_{i,j}^{(k)}}{\Delta t} = \frac{M}{4m_i m_j}\frac{|\mathbf{Q}_{i,j}^{(k+1)}|^2 + |\mathbf{Q}_{i,j}^{(k)}|^2}{|\mathbf{Q}_{i,j}^{(k+1)}|^2|\mathbf{Q}_{i,j}^{(k)}|^2}(\mathbf{P}_{i,j}^{(k+1)} + \mathbf{P}_{i,j}^{(k)}), \quad \Phi(\mathbf{Q}^{(k+1)}) = 0$$

$$\frac{\mathbf{P}_{i,j}^{(k+1)} - \mathbf{P}_{i,j}^{(k)}}{\Delta t} = \frac{\mathbf{Q}_{i,j}^{(k+1)} + \mathbf{Q}_{i,j}^{(k)}}{|\mathbf{Q}_{i,j}^{(k+1)}|^2|\mathbf{Q}_{i,j}^{(k)}|^2}\left\{\frac{M}{4m_i m_j}\left(|\mathbf{P}_{i,j}^{(k+1)}|^2 + |\mathbf{P}_{i,j}^{(k)}|^2\right) - 2m_i m_j\right\}$$

$$- \begin{bmatrix} \frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} & \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} \\ \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} & -\frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} \end{bmatrix} \mathbf{\Lambda}$$

$\mathbf{\Lambda}$, $\mathbf{\Lambda}$: Lagrangian multipliers

Regularized GTBP and BEC

Regularized GTBP

$$\frac{d}{dt}\mathbf{Q}_{i,j} = \frac{M}{m_i m_j}\frac{\mathbf{P}_{i,j}}{|\mathbf{Q}_{i,j}|^2}, \quad \Phi(\mathbf{Q}) = 0 \quad \Longleftarrow \quad \text{Fictious time } s_{i,j} : \frac{ds_{i,j}}{dt} = |\mathbf{Q}_{i,j}|^{-4}$$

$$\frac{d}{dt}\mathbf{P}_{i,j} = \frac{\mathbf{Q}_{i,j}}{|\mathbf{Q}_{i,j}|^4}\left(\frac{M}{2m_i m_j}|\mathbf{P}_{i,j}|^2 - 2m_i m_j\right) - \begin{bmatrix} Q_{i,j[2]} & Q_{i,j[1]} \\ Q_{i,j[1]} & -Q_{i,j[2]} \end{bmatrix}\lambda$$

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$$\frac{\mathbf{P}_{i,j}^{(k+1)} - \mathbf{P}_{i,j}^{(k)}}{\Delta t} = \frac{\mathbf{Q}_{i,j}^{(k+1)} + \mathbf{Q}_{i,j}^{(k)}}{|\mathbf{Q}_{i,j}^{(k+1)}|^2|\mathbf{Q}_{i,j}^{(k)}|^2}\left\{\frac{M}{4m_i m_j}\left(|\mathbf{P}_{i,j}^{(k+1)}|^2 + |\mathbf{P}_{i,j}^{(k)}|^2\right) - 2m_i m_j\right\}$$

$$- \begin{bmatrix} \frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} & \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} \\ \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} & -\frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} \end{bmatrix}\Lambda \quad \Longleftarrow$$

Discrete fictious time $s_{i,j}^{(k)}$:

$$\frac{\Delta s_{i,j}}{\Delta t} = \frac{1}{|\mathbf{Q}_{i,j}^{(k+1)}|^2|\mathbf{Q}_{i,j}^{(k)}|^2}$$

Regularized GTBP and BEC

Regularized GTBP

$$\frac{d\mathbf{Q}_{i,j}}{ds_{i,j}} = \frac{M}{m_i m_j} |\mathbf{Q}_{i,j}|^2 \mathbf{P}_{i,j}, \quad \Phi(\mathbf{Q}) = 0$$

Eliminating singularities (Regularization)

$$\frac{d\mathbf{P}_{i,j}}{ds_{i,j}} = \mathbf{Q}_{i,j} \left(\frac{M}{2 m_i m_j} |\mathbf{P}_{i,j}|^2 - 2 m_i m_j \right) - |\mathbf{Q}_{i,j}|^4 \begin{bmatrix} Q_{i,j[2]} & Q_{i,j[1]} \\ Q_{i,j[1]} & -Q_{i,j[2]} \end{bmatrix} \lambda$$

⇓ **Energy-momentum**

⇓ **integrator**

Basic Energy Conservative scheme (Gonzalez, 1999)

$$\frac{\mathbf{Q}_{i,j}^{(k+1)} - \mathbf{Q}_{i,j}^{(k)}}{\Delta s_{i,j}} = \frac{M}{4 m_i m_j} \left(|\mathbf{Q}_{i,j}^{(k+1)}|^2 + |\mathbf{Q}_{i,j}^{(k)}|^2 \right) (\mathbf{P}_{i,j}^{(k+1)} + \mathbf{P}_{i,j}^{(k)}), \quad \Phi(\mathbf{Q}^{(k+1)}) = 0$$

$$\frac{\mathbf{P}_{i,j}^{(k+1)} - \mathbf{P}_{i,j}^{(k)}}{\Delta s_{i,j}} = \left(\mathbf{Q}_{i,j}^{(k+1)} + \mathbf{Q}_{i,j}^{(k)} \right) \left\{ \frac{M}{4 m_i m_j} \left(|\mathbf{P}_{i,j}^{(k+1)}|^2 + |\mathbf{P}_{i,j}^{(k)}|^2 \right) - 2 m_i m_j \right\}$$

$$- |\mathbf{Q}_{i,j}^{(k+1)}|^2 |\mathbf{Q}_{i,j}^{(k)}|^2 \begin{bmatrix} \frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} & \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} \\ \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} & -\frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} \end{bmatrix} \Lambda$$

Eliminating singularities (Regularization)

Fault of BEC

Advantages of BEC

Analytical features

- **All conserved quantities but angular momentum are preserved.**
  - Hamiltonian
  - Linear momentum : $\sum_{i=1}^{3} \mathbf{p}_i$
  - Position of center of mass : $\sum_{i=1}^{3} m_i \mathbf{q}_i$
- **The existence of Lagrangian triangle solutions can be proved.**

Numerical features

- The errors do not appear in the case of very close encounter.

  $\Longleftarrow$ Regularization
- All masses move along a figure eight shaped orbit.

Fault of BEC

Numerical features

- **Hamiltonian drift is observed.**

  (R.I.McLachlan & M.Perlmutter, 2004)

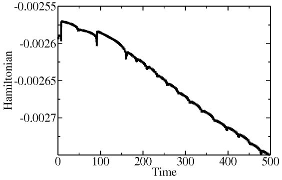


Fig. Absolute error of H

The condition number of the iteration matrix is very large.

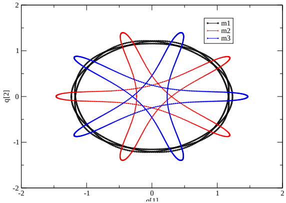
- $\Rightarrow$ **BEC destroys the long-time behaviour** which Lagrangian triangle solutions with linear stability have.
- Broucke's periodic orbits can not be reconstructed.

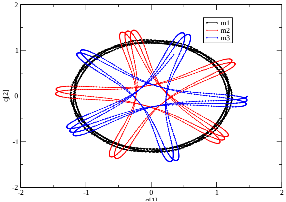
Fault of BEC

Broucke's periodic orbits $m_1 = m_2 = m_3 = \frac{1}{3}, \Delta t = 0.05$

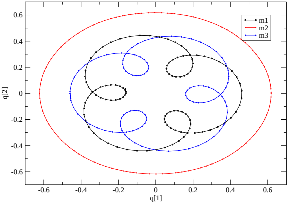
- Theoretical Orbits
- BEC $(0 \leq t \leq 25)$

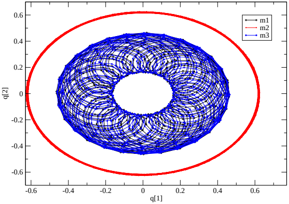
[Family A]
Commensurability ratio
$$\frac{3}{5}$$





[Family R]
Commensurability ratio
$$\frac{1}{3}$$





Constrained formulation without Lagrangian multipliers (Proposed method)

Regularized GTBP [Constrained formulation with Lagrangian multipliers]

$$\frac{d}{dt}\mathbf{Q}_{i,j} = \frac{M}{m_i m_j}\frac{\mathbf{P}_{i,j}}{|\mathbf{Q}_{i,j}|^2},\ \ \Phi(\mathbf{Q}) = 0$$

$$\frac{d}{dt}\mathbf{P}_{i,j} = \frac{\mathbf{Q}_{i,j}}{|\mathbf{Q}_{i,j}|^4}\left(\frac{M}{2m_i m_j}|\mathbf{P}_{i,j}|^2 - 2m_i m_j\right) - \begin{bmatrix} Q_{i,j[2]} & Q_{i,j[1]} \\ Q_{i,j[1]} & -Q_{i,j[2]} \end{bmatrix} \boldsymbol{\lambda}$$

Constrained formulation without Lagrangian multipliers (Proposed method)

Regularized GTBP [Constrained formulation with Lagrangian multipliers]

$$\frac{d}{dt}\mathbf{Q}_{i,j} = \frac{M}{m_i m_j}\frac{\mathbf{P}_{i,j}}{|\mathbf{Q}_{i,j}|^2}, \quad \Phi(\mathbf{Q}) = 0 \implies \frac{d^2\Phi(\mathbf{Q})}{dt^2} = 0$$

$$\frac{d}{dt}\mathbf{P}_{i,j} = \frac{\mathbf{Q}_{i,j}}{|\mathbf{Q}_{i,j}|^4}\left(\frac{M}{2m_i m_j}|\mathbf{P}_{i,j}|^2 - 2m_i m_j\right) - \begin{bmatrix} Q_{i,j[2]} & Q_{i,j[1]} \\ Q_{i,j[1]} & -Q_{i,j[2]} \end{bmatrix} \lambda$$

$\Downarrow$ Eliminating λ

Constrained formulation without Lagrangian multipliers (Proposed method)

Regularized GTBP [Constrained formulation with Lagrangian multipliers]

$$\frac{d}{dt}\mathbf{Q}_{i,j} = \frac{M}{m_i m_j}\frac{\mathbf{P}_{i,j}}{|\mathbf{Q}_{i,j}|^2},\quad \Phi(\mathbf{Q}) = 0$$

$$\frac{d}{dt}\mathbf{P}_{i,j} = \frac{\mathbf{Q}_{i,j}}{|\mathbf{Q}_{i,j}|^4}\left(\frac{M}{2m_i m_j}|\mathbf{P}_{i,j}|^2 - 2m_i m_j\right) - \begin{bmatrix} Q_{i,j[2]} & Q_{i,j[1]} \\ Q_{i,j[1]} & -Q_{i,j[2]} \end{bmatrix}\boldsymbol{\lambda}$$

⇓ Eliminating $\boldsymbol{\lambda}$

Constrained formulation without Lagrangian multipliers

$$\frac{d}{dt}\mathbf{Q}_{i,j} = \frac{M}{m_i m_j}\frac{\mathbf{P}_{i,j}}{|\mathbf{Q}_{i,j}|^2},\quad \Phi(\mathbf{Q}) = 0$$

$$f(\mathbf{P}_{1,2},\mathbf{Q}_{1,2}) = f(\mathbf{P}_{2,3},\mathbf{Q}_{2,3}),\quad f(\mathbf{P}_{2,3},\mathbf{Q}_{2,3}) = f(\mathbf{P}_{3,1},\mathbf{Q}_{3,1}),$$

where

$$f(\mathbf{P}_{i,j},\mathbf{Q}_{i,j}) = \frac{1}{|\mathbf{Q}_{i,j}|^2}\begin{bmatrix} Q_{i,j[2]} & Q_{i,j[1]} \\ Q_{i,j[1]} & -Q_{i,j[2]} \end{bmatrix}\left\{\frac{d\mathbf{P}_{i,j}}{dt} - \frac{\mathbf{Q}_{i,j}}{|\mathbf{Q}_{i,j}|^4}\left(\frac{M}{2m_i m_j}|\mathbf{P}_{i,j}|^2 - 2m_i m_j\right)\right\}$$

□ : Two-body problem

Constrained formulation without Lagrangian multipliers (Proposed method)

BEC

$\Downarrow$ **Eliminating λ**

Proposed method (d-GTBP)

$$\frac{\mathbf{Q}_{i,j}^{(k+1)} - \mathbf{Q}_{i,j}^{(k)}}{\Delta t} = \frac{M}{4m_i m_j}\frac{|\mathbf{Q}_{i,j}^{(k+1)}|^2 + |\mathbf{Q}_{i,j}^{(k)}|^2}{|\mathbf{Q}_{i,j}^{(k+1)}|^2|\mathbf{Q}_{i,j}^{(k)}|^2}(\mathbf{P}_{i,j}^{(k+1)} + \mathbf{P}_{i,j}^{(k)}), \quad \Phi(\mathbf{Q}^{(k+1)}) = 0$$

$$F(1,2) = F(2,3) = F(3,1),$$

where

$$F(i,j) \equiv \frac{1}{|\mathbf{Q}_{i,j}^{(k+1)}|^2 + |\mathbf{Q}_{i,j}^{(k)}|^2}\begin{bmatrix} \frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} & \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} \\ \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} & -\frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} \end{bmatrix}$$

$$\left\{\frac{\mathbf{P}_{i,j}^{(k+1)} - \mathbf{P}_{i,j}^{(k)}}{\Delta t} - \frac{\mathbf{Q}_{i,j}^{(k+1)} + \mathbf{Q}_{i,j}^{(k)}}{|\mathbf{Q}_{i,j}^{(k+1)}|^2|\mathbf{Q}_{i,j}^{(k)}|^2}\left(\frac{M}{8m_i m_j}\left(|\mathbf{P}_{i,j}^{(k+1)}|^2 + |\mathbf{P}_{i,j}^{(k)}|^2\right) - m_i m_j\right)\right\}$$

□ : A conservative discrete-time two-body problem (2002,2004, Y.M and Y. Nakamura)

Numerical results for d-GTBP

Conserved quantities

Elliptic Lagrangian triangular solution with linear stability

$(\beta = 0.14,\ e = 0.54)$

$\Delta t = 0.01,\ 0 \leq t \leq 10000$

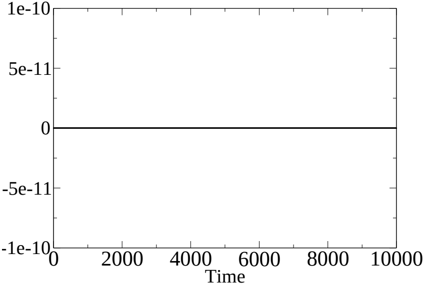


Fig. Error of Hamiltonian

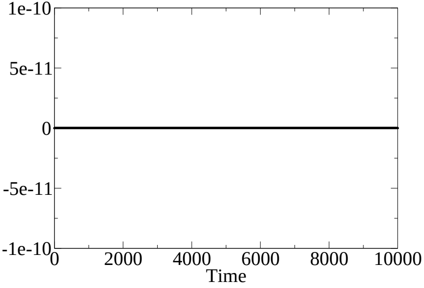


Fig. Error of angular momentum

Conserved quantities

Elliptic Lagrangian triangular solution with linear stability
$(\beta = 0.14,\ e = 0.54)$
$\Delta t = 0.01,\ 0 \le t \le 10000$

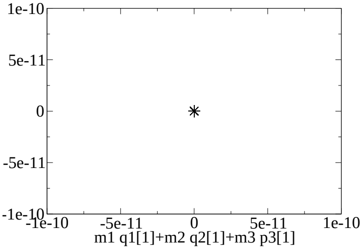


Fig. Error of the position of center of mass

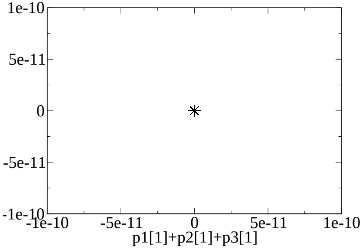
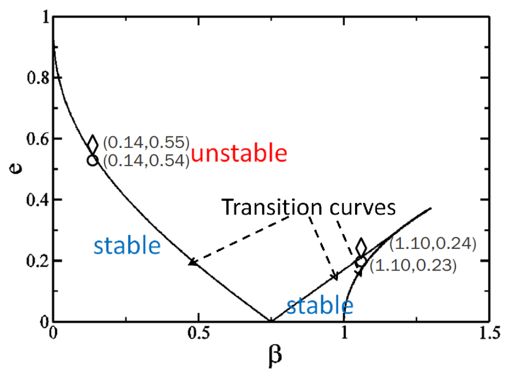


Fig. Error of linear momentum

Numerical results for d-GTBP

Elliptic Lagrangian triangle solutions

- **Linear stability of the elliptic Lagrangian triangle solutions (J.M.A. Danby, 1964; G.E. Roberts, 2000; ....)**
  - **the mass parameter :** $\beta = \dfrac{27(m_1 m_2 + m_2 m_3 + m_3 m_1)}{M}$
  - **the eccentricity of the orbit :** e



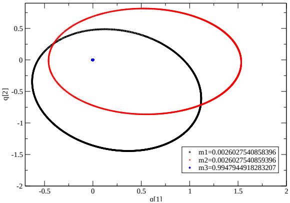
Numerical results for d-GTBP

Elliptic Lagrangian triangle solutions $\Delta t = 0.01,\ 0 \le t \le 10000$

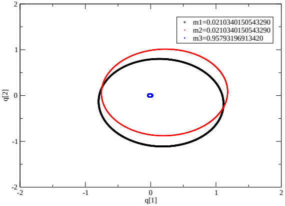
$\Rightarrow$ Proposed method (d-GTBP) can draw both stable and unstable orbits correctly.

- **Orbits with linear stability** in theory

$\beta = 0.14$
$e = 0.54$

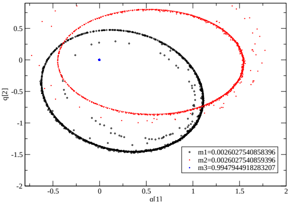


$\beta = 1.10$
$e = 0.23$



- **Orbits with linear instability** in theory

$\beta = 0.14$
$e = 0.55$



$\beta = 1.10$
$e = 0.24$

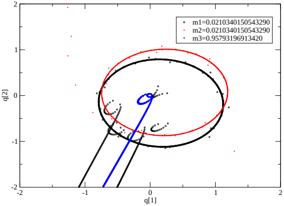


Figure-eight choreography

$\Delta t = 0.01,\ 0 \le t \le 10000$ (1580 revolutions)

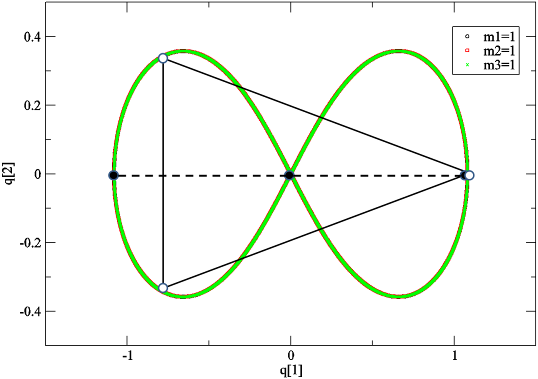


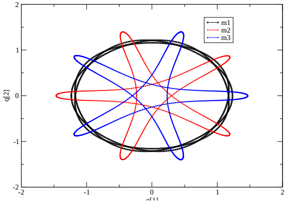
Fig. Figure-eight choreography

Numerical Results for d-GTBP

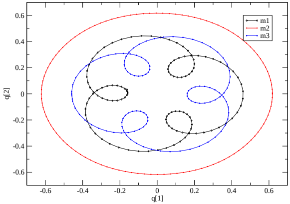
Broucke's periodic orbits $m_1 = m_2 = m_3 = \frac{1}{3}, \Delta t = 0.05$

- Proposed method (d-GTBP) $(0 \leq t \leq 100)$

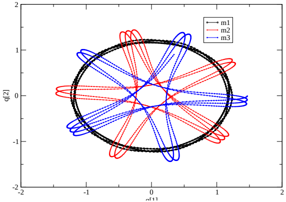
[Family A]
Commensurability ratio $\frac{3}{5}$
11 revolutions

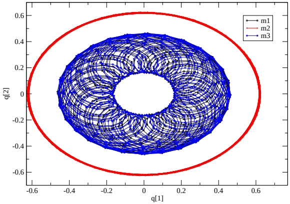


[Family R]
Commensurability ratio $\frac{1}{3}$
18 revolutions



- BEC $(0 \leq t \leq 25)$





BEC vs d-GTBP

Fault of BEC

Numerical features

- Hamiltonian drift is observed.
- BEC destroys the long-time behaviour which Lagrangian triangle solutions with linear stability have.
- Brocke's periodic orbits **can not** be reproduced.

Advantages of BEC

Numerical features

- The errors do not appear in the case of very close encounter.
- All masses move along a figure eight shaped orbit.

⇒

Advantages of d-GTBP

Numerical features

- All conserved quantities are kept constant.
- Of the orbits corresponding to Lagrangian triangle solutions,
  - **linearly stable** ones are **closed** ,
  - **linearly unstable** ones are **not closed**.
- Brocke's periodic orbits **can** be reproduced.

- The errors do not appear in the case of very close encounter.
- All masses move along a figure eight shaped orbit.

Conclusions for d-GTBP

- d-GTBP (= a proposed method for GTBP) has the following advantages.
  - The errors do not appear in the case of very close encounter.
  - **Theoretically,** all conserved quantities **but angular momentum** are kept constant.
  - **For stable periodic orbits, All conserved quantities** are kept constant.
  - A lot of orbits corresponding to particular solutions can be drawn.

Conclusions for d-GTBP

- d-GTBP (= a proposed method for GTBP) has the following advantages.
  $\Longrightarrow$ d-GTBP can draw orbits more correctly than symplectic integrator and energy-momentum integrator.
  - The errors do not appear in the case of very close encounter.
  - **Theoretically,** all conserved quantities **but angular momentum** are kept constant.
  - **For stable periodic orbits, All conserved quantities** are kept constant.
  - A lot of orbits corresponding to particular solutions can be drawn.

Proposed method for RTBP (d-RTBP)

Proposed method (d-RTBP) $\overset{m_3 \to 0}{\Longleftarrow\Longleftarrow}$ a discrete eq. $\overset{\mathbf{V}_{i,j} = \mathbf{P}_{i,j}/m_3,\ (i,j) = (2,3),(3,1)}{\Longleftarrow\Longleftarrow\Longleftarrow\Longleftarrow\Longleftarrow\Longleftarrow\Longleftarrow\Longleftarrow\Longleftarrow\Longleftarrow}$ d-GTBP

$$\begin{cases} \dfrac{\mathbf{Q}_{1,2}^{(k+1)} - \mathbf{Q}_{1,2}^{(k)}}{\Delta t} = \dfrac{M}{4m_1 m_2} \dfrac{|\mathbf{Q}_{1,2}^{(k+1)}|^2 + |\mathbf{Q}_{1,2}^{(k)}|^2}{|\mathbf{Q}_{1,2}^{(k+1)}|^2 |\mathbf{Q}_{1,2}^{(k)}|^2} (\mathbf{P}_{1,2}^{(k+1)} + \mathbf{P}_{1,2}^{(k)}), \\ \dfrac{\mathbf{P}_{1,2}^{(k+1)} - \mathbf{P}_{1,2}^{(k)}}{\Delta t} = \left(\dfrac{M}{4m_1 m_2} (|\mathbf{P}_{1,2}^{(k+1)}|^2 + |\mathbf{P}_{1,2}^{(k)}|^2) - 2m_1 m_2 \right) \dfrac{\mathbf{Q}_{1,2}^{(k+1)} + \mathbf{Q}_{1,2}^{(k)}}{|\mathbf{Q}_{1,2}^{(k+1)}|^2 |\mathbf{Q}_{1,2}^{(k)}|^2}, \end{cases}$$

□ : a conservative discrete-time two-body problem

$$\begin{cases} \dfrac{\mathbf{Q}_{i,j}^{(k+1)} - \mathbf{Q}_{i,j}^{(k)}}{\Delta t} = \dfrac{M m_3}{4m_i m_j} \dfrac{|\mathbf{Q}_{i,j}^{(k+1)}|^2 + |\mathbf{Q}_{i,j}^{(k)}|^2}{|\mathbf{Q}_{i,j}^{(k+1)}|^2 |\mathbf{Q}_{i,j}^{(k)}|^2} (\mathbf{V}_{i,j}^{(k+1)} + \mathbf{V}_{i,j}^{(k)}), \quad (i,j) = (2,3),(3,1), \\ \mathbf{F}(\mathbf{V}_{2,3}^{(k+1)}, \mathbf{Q}_{2,3}^{(k+1)}, \mathbf{V}_{2,3}^{(k)}, \mathbf{Q}_{2,3}^{(k)}) = \mathbf{F}(\mathbf{V}_{3,1}^{(k+1)}, \mathbf{Q}_{3,1}^{(k+1)}, \mathbf{V}_{3,1}^{(k)}, \mathbf{Q}_{3,1}^{(k)}) \end{cases}$$

where

$$\mathbf{F}(\mathbf{V}_{i,j}^{(k+1)}, \mathbf{Q}_{i,j}^{(k+1)}, \mathbf{V}_{i,j}^{(k)}, \mathbf{Q}_{i,j}^{(k)}) \equiv \frac{1}{|\mathbf{Q}_{1,2}^{(k+1)}|^2 |\mathbf{Q}_{1,2}^{(k)}|^2} \begin{bmatrix} \frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} & \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} \\ \frac{Q_{i,j[1]}^{(k+1)} + Q_{i,j[1]}^{(k)}}{2} & -\frac{Q_{i,j[2]}^{(k+1)} + Q_{i,j[2]}^{(k)}}{2} \end{bmatrix}$$

$$\left\{ \frac{\mathbf{V}_{i,j}^{(k+1)} - \mathbf{V}_{i,j}^{(k)}}{\Delta t} - \frac{\mathbf{Q}_{i,j}^{(k+1)} + \mathbf{Q}_{i,j}^{(k)}}{|\mathbf{Q}_{i,j}^{(k+1)}|^2 |\mathbf{Q}_{i,j}^{(k)}|^2} \left(\frac{M m_3}{4m_i m_j} (|\mathbf{V}_{i,j}^{(k+1)}|^2 + |\mathbf{V}_{i,j}^{(k)}|^2) - 2\frac{m_i m_j}{m_3} \right) \right\}$$

Conserved quantities

Elliptic Lagrangian triangular solution with linear stability

$\left(\nu = \frac{m_2}{M} = 0.19,\ e = 0.19\right)$

$\Delta t = 0.01,\ 0 \le t \le 10000$

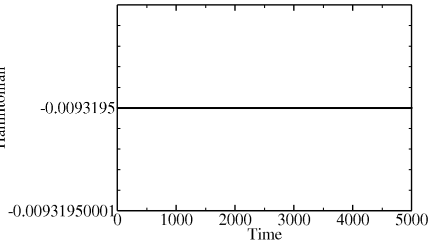


Fig. Error of Hamiltonian

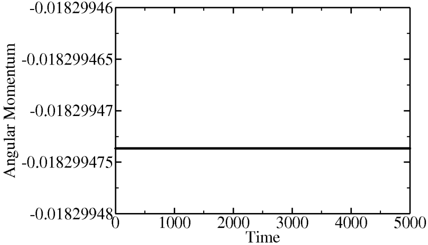


Fig. Error of angular momentum

Numerical results for d-RTBP

Conserved quantities

Elliptic Lagrangian triangular solution with linear stability

$\left(\nu = \frac{m_2}{M} = 0.19,\ e = 0.19\right)$

$\Delta t = 0.01,\ 0 \leq t \leq 10000$

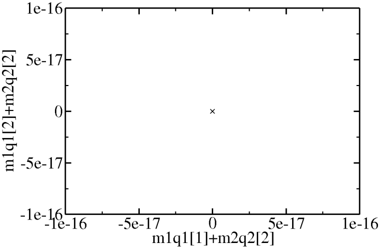


Fig. Error of the position of center of mass

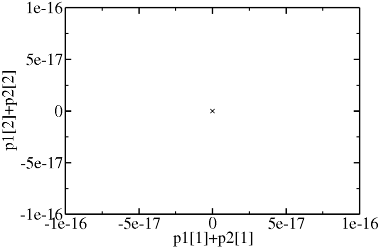


Fig. Error of linear momentum

Numerical results for d-RTBP

Conserved quantities

Elliptic Lagrangian triangular solution with linear stability

$\left(\nu = \frac{m_2}{M} = 0.19,\ e = 0.19\right)$

$\Delta t = 0.01,\ 0 \leq t \leq 10000$

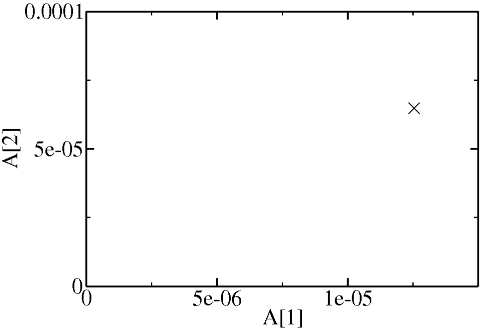


Fig. Error of Laplace vector

Theoretically and numerically,

- Hamiltonian
- Linear momentum
- Angular momentum
- Center of mass
- Laplace vector

are **the conserved quantities of a three-body problem** and all preserved.

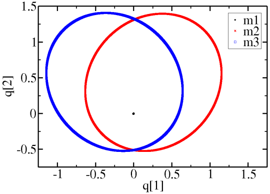
Numerical results for d-RTBP

Elliptic Lagrangian triangle solutions

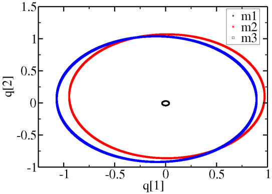
$\Delta t = 0.01, \ 0 \leq t \leq 10000$

- Orbits with linear stability

$\nu = 0.006$
$e = 0.51$

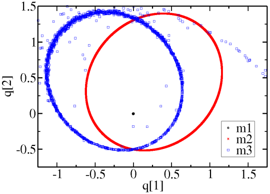


$\nu = 0.035$
$e = 0.11$

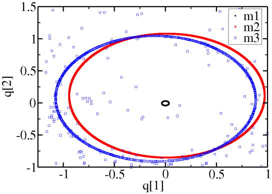


- Orbits with linear instability

$\nu = 0.006$
$e = 0.52$



$\nu = 0.035$
$e = 0.12$



Numerical results for d-RTBP

Broucke's periodic orbits (Broucke,1968)

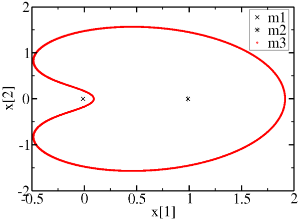
$\Delta t = 0.01,\ 0 \leq t \leq 10000$

- **Linear stable orbits in the family J_1**

J1-20

more than

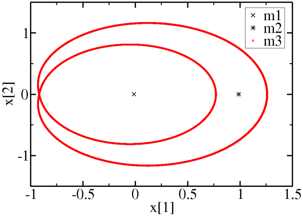
1500 revolutions



J1-91

more than

1500 revolutions

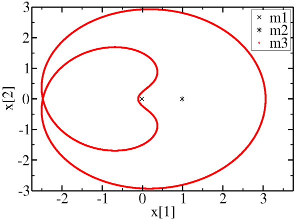


- **Linear stable orbits in the family A_1**

A1-226

more than

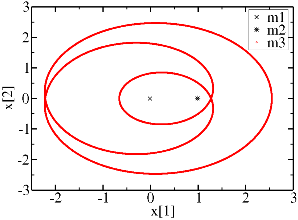
780 revolutions



A1-310

more than

780 revolutions



Numerical results for d-RTBP

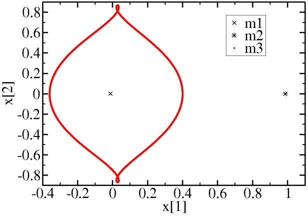
Broucke's periodic orbits (Broucke,1968) $\Delta t = 0.01,\ 0 \leq t \leq 10000$

- Linear stable orbits in the family BD

BD-61

more than

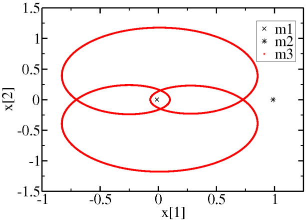
1500 revolutions



BD-71

more than

1500 revolutions

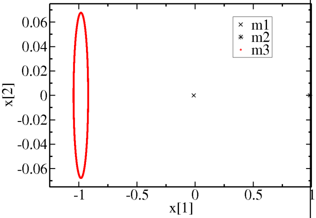


- Linear stable orbits in the family H

H1-11

more than

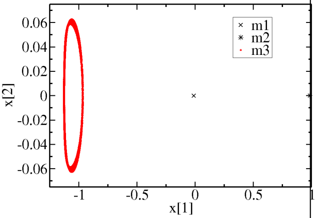
8800 revolutions



H2-119

more than

4050 revolutions



Numerical Feature of d-RTBP

- d-RTBP (= proposed method for RTBP) has the following advantages.
  - The errors do not appear in the case of very close encounter (cf. A1-226, BD-71).
  - All conserved quantities are kept constant.
  - Many linear stable orbits can be drawn.

Numerical Feature of d-RTBP

- d-RTBP (= proposed method for RTBP) has the following advantages.
  $\Longrightarrow$ Proposed method can draw orbits more correctly than symplectic integrator and energy-momentum integrator.
  - The errors do not appear in the case of very close encounter (cf. A1-226, BD-71).
  - All conserved quantities are kept constant.
  - Many linear stable orbits can be drawn.

Lagrangian Equilibrium Points in RTBP

For simplicity, m_1, m_2 in circular orbits.

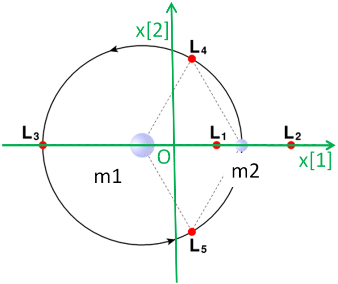


Fig. Lagrangian Equilibrium Points in the rotating barycentric coordinate system

L_1, L_2, L_3: **Collinear Equilibrium Points**

$\Longrightarrow$ Linear Unstable

L_4, L_5: **Triangular Equilibrium Points**

$\Longrightarrow$ Linear Stable

LEPs in the rotating LC variables

$\underline{L_1}$:

$$\mathbf{x}_{1,2} = \left(r^{\frac{1}{2}}, -r^{\frac{1}{2}}\right), \quad \mathbf{x}_{2,3} = \left(\underline{x^{(L_1)}_{2,3[1]}}, -\underline{x^{(L_1)}_{2,3[1]}}\right),$$

$$\mathbf{x}_{3,1} = \left(x^{(L_1)}_{3,1[1]}, x^{(L_1)}_{3,1[1]}\right),$$

$$\mathbf{Y}_{1,2} = \left(-\nu(1-\nu)m^{\frac{3}{2}}, -\nu(1-\nu)m^{\frac{3}{2}}\right),$$

$$\mathbf{W}_{2,3} = \left(-\nu m^{\frac{1}{2}} r^{-\frac{3}{2}} \left(\underline{x^{(L_1)}_{2,3[1]}}\right)^3, -\nu m^{\frac{1}{2}} r^{-\frac{3}{2}} \left(\underline{x^{(L_1)}_{2,3[1]}}\right)^3\right),$$

$$\mathbf{W}_{3,1} = \begin{pmatrix} (1-\nu)m^{1/2} r^{-\frac{3}{2}} \left(x^{(L_1)}_{3,1[1]}\right)^3 \\ -(1-\nu)m^{1/2} r^{-\frac{3}{2}} \left(x^{(L_1)}_{3,1[1]}\right)^3 \end{pmatrix}^{\top}.$$

where $\left(x^{(L_1)}_{3,1}\right)^2 = \left(x^{(L_1)}_{2,3}\right)^2 + r$.

$\underline{L_4}$:

$$\mathbf{x}_{1,2} = \left(r^{\frac{1}{2}}, -r^{\frac{1}{2}}\right), \quad \mathbf{x}_{2,3} = \left(\frac{\sqrt{3}-1}{2} r^{\frac{1}{2}}, \frac{\sqrt{3}+1}{2} r^{\frac{1}{2}}\right),$$

$$\mathbf{x}_{3,1} = \left(\frac{\sqrt{3}+1}{2} r^{\frac{1}{2}}, \frac{\sqrt{3}-1}{2} r^{\frac{1}{2}}\right),$$

$$\mathbf{Y}_{1,2} = \left(-\nu(1-\nu)m^{\frac{3}{2}}, -\nu(1-\nu)m^{\frac{3}{2}}\right),$$

$$\mathbf{W}_{2,3} = \left(\frac{\sqrt{3}+1}{2} \nu m^{\frac{1}{2}}, -\frac{\sqrt{3}-1}{2} \nu m^{\frac{1}{2}}\right),$$

$$\mathbf{W}_{3,1} = \left(\frac{\sqrt{3}-1}{2}(1-\nu)m^{\frac{1}{2}}, \frac{\sqrt{3}+1}{2}(1-\nu)m^{\frac{1}{2}}\right).$$

Relative motion of primaries in d-RTBP

Hereafter, for simplicity, the primaries m_1 and m_2 are supposed to be moving in **circular orbits.**

Relative Motion of Primaries in **d-RTBP** in the barycentric coordinate system

$$\left[Q^{(k)}_{1,2[1]}, Q^{(k)}_{1,2[2]}\right]^\top = r^{\frac{1}{2}} \left[\cos(\frac{\Omega}{2}k\Delta t), \sin(\frac{\Omega}{2}k\Delta t)\right]^\top,$$

Discrete analogue of Kepler's 3rd law : $\frac{4}{\Delta t}\tan\frac{\Omega\Delta t}{4} = \sqrt{\frac{M}{r^3}}$

$$\Downarrow \ \Delta t \to 0$$

Relative motion of primaries in **RTBP** in the barycentric coordinate system

$$\left[Q^{(k)}_{1,2[1]}, Q^{(k)}_{1,2[2]}\right]^\top = r^{\frac{1}{2}} \left[\cos(\frac{\omega}{2}k\Delta t), \sin(\frac{\omega}{2}k\Delta t)\right]^\top,$$

Kepler's 3rd law : $\omega = \sqrt{\frac{M}{r^3}}$

d-RTBP in the rotating coordinate system

Introduce a rotating coordinate system with an uniform angular velocity $\boldsymbol{\Omega}$.

d-RTBP in the barycentric coordinate system ($O - Q_{[1]}Q_{[2]}$ system)

$\Downarrow \; m_1 = (1-\nu)m, m_2 = \nu m$

d-RTBP in the rotating coordinate system ($O - X_{[1]}X_{[2]}$ system)

$$\mathbf{X}_{1,2} = \left(r^{1/2}, -r^{1/2}\right), \mathbf{Y}_{1,2} = \left(-\nu(1-\nu)m^{3/2}, -\nu(1-\nu)m^{3/2}\right) \; : \; \text{Fixed Points}$$

$$\begin{cases} \dfrac{(1-z^2)X^{(k+1)}_{2,3[1]} - (1+z^2)X^{(k)}_{2,3[1]}}{\Delta t} = -\dfrac{2z}{\Delta t}X^{(k+1)}_{2,3[2]} + \dfrac{1}{4\nu}\dfrac{|\mathbf{X}^{(k+1)}_{2,3}|^2 + |\mathbf{X}^{(k)}_{2,3}|^2}{|\mathbf{X}^{(k+1)}_{2,3}|^2|\mathbf{X}^{(k)}_{2,3}|^2}A_1(\mathbf{W}_{2,3}), \\ \dfrac{(1-z^2)X^{(k+1)}_{2,3[2]} - (1+z^2)X^{(k)}_{2,3[2]}}{\Delta t} = \dfrac{2z}{\Delta t}X^{(k+1)}_{2,3[1]} + \dfrac{1}{4\nu}\dfrac{|\mathbf{X}^{(k+1)}_{2,3}|^2 + |\mathbf{X}^{(k)}_{2,3}|^2}{|\mathbf{X}^{(k+1)}_{2,3}|^2|\mathbf{X}^{(k)}_{2,3}|^2}A_2(\mathbf{W}_{2,3}), \end{cases}$$

$$\begin{cases} \dfrac{(1-z^2)X^{(k+1)}_{3,1[1]} - (1+z^2)X^{(k)}_{3,1[1]}}{\Delta t} = -\dfrac{2z}{\Delta t}X^{(k+1)}_{3,1[2]} + \dfrac{1}{4(1-\nu)}\dfrac{|\mathbf{X}^{(k+1)}_{3,1}|^2 + |\mathbf{X}^{(k)}_{3,1}|^2}{|\mathbf{X}^{(k+1)}_{3,1}|^2|\mathbf{X}^{(k)}_{3,1}|^2}A_1(\mathbf{W}_{3,1}), \\ \dfrac{(1-z^2)X^{(k+1)}_{3,1[2]} - (1+z^2)X^{(k)}_{3,1[2]}}{\Delta t} = \dfrac{2z}{\Delta t}X^{(k+1)}_{3,1[1]} + \dfrac{1}{4(1-\nu)}\dfrac{|\mathbf{X}^{(k+1)}_{3,1}|^2 + |\mathbf{X}^{(k)}_{3,1}|^2}{|\mathbf{X}^{(k+1)}_{3,1}|^2|\mathbf{X}^{(k)}_{3,1}|^2}A_2(\mathbf{W}_{3,1}), \end{cases}$$

where $z = \tan\dfrac{\Omega\Delta t}{4}$, $A_1(\mathbf{x}) = (1-z^2)x^{(k+1)}_{[1]} + 2zx^{(k+1)}_{[2]} + (1+z^2)x^{(k)}_{[1]}$,

$A_2(\mathbf{x}) = -2zx^{(k+1)}_{[1]} + (1-z^2)x^{(k+1)}_{[2]} + (1+z^2)x^{(k)}_{[2]}$.

d-RTBP in the rotating coordinate system

d-RTBP in the rotating coordinate system ($O - X_{[1]}X_{[2]}$ system)

$$\begin{cases} X_{1,2[1]}^{(k+1)} X_{1,2[2]}^{(k+1)} + X_{2,3[1]}^{(k+1)} X_{2,3[2]}^{(k+1)} + X_{3,1[1]}^{(k+1)} X_{3,1[2]}^{(k+1)} = 0, \\ \frac{1}{2}\Big((X_{1,2[1]}^{(k+1)})^2 - (X_{1,2[2]}^{(k+1)})^2\Big) + \frac{1}{2}\Big((X_{2,3[1]}^{(k+1)})^2 - (X_{2,3[2]}^{(k+1)})^2\Big) + \frac{1}{2}\Big((X_{3,1[1]}^{(k+1)})^2 - (X_{3,1[2]}^{(k+1)})^2\Big) = 0, \end{cases}$$

$$G_{2,3[1]} = G_{3,1[1]}, \; G_{2,3[2]} = G_{3,1[2]},$$

where

$$\begin{aligned} G_{2,3[1]} = B(\mathbf{x})\Big(& \frac{1}{2\Delta t} \Big(A_1(\mathbf{x})(-2zy_{[1]}^{(k+1)} + (1-z^2)y_{[2]}^{(k+1)} - (1+z^2)y_{[2]}^{(k)}) \\ & + A_2(\mathbf{x})((1-z^2)y_{[1]}^{(k+1)} + 2zy_{[2]}^{(k+1)} - (1+z^2)y_{[1]}^{(k)})\Big) \\ & - \frac{2}{|\mathbf{x}^{(k+1)}|^2|\mathbf{x}^{(k)}|^2} \left(\frac{1}{8\nu}(|\mathbf{y}^{(k+1)}|^2 + |\mathbf{y}^{(k)}|^2) - \nu m \right) A_1(\mathbf{x})A_2(\mathbf{x})\Big), \end{aligned}$$

$$\begin{aligned} G_{3,1[1]} = B(\mathbf{x})\Big(& \frac{1}{2\Delta t} \Big(A_1(\mathbf{x})(-2zy_{[1]}^{(k+1)} + (1-z^2)y_{[2]}^{(k+1)} - (1+z^2)y_{[2]}^{(k)}) \\ & + A_2(\mathbf{x})((1-z^2)y_{[1]}^{(k+1)} + 2zy_{[2]}^{(k+1)} - (1+z^2)y_{[1]}^{(k)})\Big) \\ & - \frac{2}{|\mathbf{x}^{(k+1)}|^2|\mathbf{x}^{(k)}|^2} \left(\frac{1}{8(1-\nu)}(|\mathbf{y}^{(k+1)}|^2 + |\mathbf{y}^{(k)}|^2) - (1-\nu) m \right) A_1(\mathbf{x})A_2(\mathbf{x})\Big), \end{aligned}$$

d-RTBP in the rotating coordinate system ($\mathrm{O}-X_{[1]}X_{[2]}$ system)

$$G_{2,3[2]}$$

$$= B(\mathrm{x})\Bigg(\frac{1}{2\Delta t}\Big(A_1(\mathrm{x})\big((1-z^2)y_{[1]}^{(k+1)} + 2zy_{[2]}^{(k+1)} - (1+z^2)y_{[1]}^{(k)}\big) - A_2(\mathrm{x})\big(-2zy_{[1]}^{(k+1)} + (1-z^2)y_{[2]}^{(k+1)} - (1+z^2)y_{[2]}^{(k)}\big)\Big) - \frac{1}{|\mathrm{x}^{(k+1)}|^2|\mathrm{x}^{(k)}|^2}\left(\frac{1}{8\nu}(|\mathrm{y}^{(k+1)}|^2 + |\mathrm{y}^{(k)}|^2) - \nu m\right)\left((A_1(\mathrm{x}))^2 - (A_2(\mathrm{x}))^2\right)\Bigg),$$

$$G_{3,1[2]}$$

$$= B(\mathrm{x})\Bigg(\frac{1}{2\Delta t}\Big(A_1(\mathrm{x})\big((1-z^2)y_{[1]}^{(k+1)} + 2zy_{[2]}^{(k+1)} - (1+z^2)y_{[1]}^{(k)}\big) - A_2(\mathrm{x})\big(-2zy_{[1]}^{(k+1)} + (1-z^2)y_{[2]}^{(k+1)} - (1+z^2)y_{[2]}^{(k)}\big)\Big) - \frac{1}{|\mathrm{x}^{(k+1)}|^2|\mathrm{x}^{(k)}|^2}\left(\frac{1}{8(1-\nu)}(|\mathrm{y}^{(k+1)}|^2 + |\mathrm{y}^{(k)}|^2) - (1-\nu)m\right)\left((A_1(\mathrm{x}))^2 - (A_2(\mathrm{x}))^2\right)\Bigg),$$

$$B(\mathrm{x}) = (1+z^2)^{-1} \times \Big((|\mathrm{x}^{(k+1)}|^2 + |\mathrm{x}^{(k)}|^2) + 2(1-z^2)(x_{[1]}^{(k+1)}x_{[1]}^{(k)} + x_{[2]}^{(k+1)}x_{[2]}^{(k)}) - 4z(x_{[1]}^{(k+1)}x_{[2]}^{(k)} - x_{[2]}^{(k+1)}x_{[1]}^{(k)})\Big)^{-1}.$$

Outline for existence proof of LEPs

- Assume that the d-RTBP has the same LEPs as the original RTBP in the rotating coordinate system at two discrete-time $t^{(k)}$ and $t^{(k+1)}$, namely, $\mathbf{X}_{i,j}^{(k)} = \mathbf{X}_{i,j}^{(k+1)} = \mathbf{X}_{i,j},\ (i,j) = (1,2), (2,3), (3,1);\ \mathbf{Y}_{1,2}^{(k)} = \mathbf{Y}_{1,2}^{(k+1)} = \mathbf{Y}_{i,j};$ $\mathbf{W}_{i,j}^{(k)} = \mathbf{W}_{i,j}^{(k+1)} = \mathbf{W}_{i,j},\ (i,j) = (2,3), (3,1)$

Example 1 L_1 : **Collinear Equilibrium Points**

$$\mathbf{X}_{1,2} = \left(r^{\frac{1}{2}}, -r^{\frac{1}{2}}\right),\ \mathbf{X}_{2,3} = \left(\underline{X_{2,3[1]}^{(L_1)}}, -\underline{X_{2,3[1]}^{(L_1)}}\right),\ \mathbf{X}_{3,1} = \left(X_{3,1[1]}^{(L_1)}, X_{3,1[1]}^{(L_1)}\right),$$

$$\mathbf{Y}_{1,2} = \left(-\nu(1-\nu)m^{\frac{3}{2}}, -\nu(1-\nu)m^{\frac{3}{2}}\right), \mathbf{W}_{2,3} = \left(-\nu m^{\frac{1}{2}} r^{-\frac{3}{2}} (\underline{X_{2,3[1]}^{(L_2)}})^3, -\nu m^{\frac{1}{2}} r^{-\frac{3}{2}} (\underline{X_{2,3[1]}^{(L_1)}})^3\right),$$

$$\mathbf{W}_{3,1} = \left((1-\nu)m^{\frac{1}{2}} r^{-\frac{3}{2}} (X_{3,1[1]}^{(L_1)})^3, -(1-\nu)m^{\frac{1}{2}} r^{-\frac{3}{2}} (X_{3,1[1]}^{(L_1)})^3\right).$$

where $(X_{3,1[1]}^{(L_1)})^2 = (X_{2,3[1]}^{(L_1)})^2 + r.$

Example 2 L_4 : **Triangular Equilibrium Points**

$$\mathbf{X}_{1,2} = (r^{\frac{1}{2}}, -r^{\frac{1}{2}}),\ \mathbf{X}_{2,3} = \left(\frac{\sqrt{3}-1}{2} r^{\frac{1}{2}}, \frac{\sqrt{3}+1}{2} r^{\frac{1}{2}}\right),\ \mathbf{X}_{3,1} = \left(\frac{\sqrt{3}+1}{2} r^{\frac{1}{2}}, \frac{\sqrt{3}-1}{2} r^{\frac{1}{2}}\right),$$

$$\mathbf{Y}_{1,2} = \left(-\nu(1-\nu)m^{\frac{3}{2}}, -\nu(1-\nu)m^{\frac{3}{2}}\right), \mathbf{W}_{2,3} = \left(\frac{\sqrt{3}+1}{2}\nu m^{\frac{1}{2}}, -\frac{\sqrt{3}-1}{2}\nu m^{\frac{1}{2}}\right),$$

$$\mathbf{W}_{3,1} = \left(\frac{\sqrt{3}-1}{2}(1-\nu)m^{\frac{1}{2}}, \frac{\sqrt{3}+1}{2}(1-\nu)m^{\frac{1}{2}}\right).$$

Outline for existence proof of LEPs

- Substitute
  $\mathbf{X}_{i,j}^{(k)} = \mathbf{X}_{i,j}^{(k+1)} = \mathbf{X}_{i,j},\ (i,j) = (1,2),(2,3),(3,1);\ \mathbf{Y}_{1,2}^{(k)} = \mathbf{Y}_{1,2}^{(k+1)} = \mathbf{Y}_{i,j};$
  $\mathbf{W}_{i,j}^{(k)} = \mathbf{W}_{i,j}^{(k+1)} = \mathbf{W}_{i,j},\ (i,j) = (2,3),(3,1)$ into the d-RTBP in the rotating coordinate system.

○ **In the case of L_1, L_2, L_3**

The only one quatic equation for $X_{2,3[1]}^{(L_i)}$ is given by substitution.
This equation is the same as the equation for the original RTBP .

Example 1 L_1 : **Collinear Equilibrium Points**

$$(X_{2,3[1]}^{(L_1)})^{10} + (3-\nu)r(X_{2,3[1]}^{(L_1)})^{8} + (3-\nu)r^2(X_{2,3[1]}^{(L_1)})^{6} - \nu r^3(X_{2,3[1]}^{(L_1)})^{4} - 2\nu r^4(X_{2,3[1]}^{(L_1)})^{2} - \nu r^5 = 0.$$

○ **In the case of L_4, L_5**

We can show that each equilibrium solution satisfy the d-RTBP **by substitution**.

Conclusion : The d-RTBP has **the same Lagrangian equilibrium points as the original RTBP** in a rotating barycenric coordinate system.

Linear Stablity Condition of triangular equilibrium points in RTBP

The substitution of displacements from the equilibrium point L_4

$\mathbf{X}_{2,3} \equiv \mathbf{X}_{2,3}^{(L_4)} + \Delta\mathbf{X}_{2,3}$, $\mathbf{X}_{3,1} \equiv \mathbf{X}_{3,1}^{(L_4)} + \Delta\mathbf{X}_{3,1}$, $\mathbf{W}_{2,3} \equiv \mathbf{W}_{2,3}^{(L_4)} + \Delta\mathbf{W}_{2,3}$,

$\mathbf{W}_{3,1} \equiv \mathbf{W}_{3,1}^{(L_4)} + \Delta\mathbf{W}_{3,1}$ (Note $\Delta\mathbf{X}_{1,2} = \Delta\mathbf{Y}_{1,2} = 0.$)

into RTBP.

⇓

Linear approximation of RTBP

⇓

Characteristic eq. for linear approximation of RTBP ⟸ **Algebraically solvable !**

(Gascheau, Routh, Szebehely, Roy, ...)

$$4r^6\lambda^4 + 4mr^3\lambda^2 + 27m^2\nu(1-\nu) = 0.$$

Eigenvalues ≡ The solution of characteristic eq.

$$\lambda = \pm\sqrt{\frac{M}{2r^3}\left(-1+\sqrt{1-27\nu+27\nu^2}\right)},\ \pm\sqrt{\frac{M}{2r^3}\left(-1-\sqrt{1-27\nu+27\nu^2}\right)}$$

Linear Stability Condition ≡ The condition that **the real part of λ is negative**.

$1-27\nu+27\nu^2 > 0$, namely $0 \le \nu < \dfrac{1}{2} - \dfrac{\sqrt{69}}{18}$

The substitution of displacements from the equilibrium point L_4
$\mathbf{X}_{2,3}^{(j)} \equiv \mathbf{X}_{2,3}^{(L_4)} + \Delta\mathbf{X}_{2,3}^{(j)}$, $\mathbf{X}_{3,1}^{(j)} \equiv \mathbf{X}_{3,1}^{(L_4)} + \Delta\mathbf{X}_{3,1}^{(j)}$, $\mathbf{W}_{2,3}^{(j)} \equiv \mathbf{W}_{2,3}^{(L_4)} + \Delta\mathbf{W}_{2,3}^{(j)}$,
$\mathbf{W}_{3,1}^{(j)} \equiv \mathbf{W}_{3,1}^{(L_4)} + \Delta\mathbf{W}_{3,1}^{(j)}$ (Note $\Delta\mathbf{X}_{1,2}^{(j)} = \Delta\mathbf{Y}_{1,2}^{(j)} = 0$), j=k,k+1
into d-RTBP.

⇓

Linear approximation of d-RTBP

⇓

Characteristic eq. for the linear approximation of d-RTBP

$$e_0\lambda^6 + e_1\lambda^5 + e_2\lambda^4 + e_3\lambda^3 + e_2\lambda^2 + e_1\lambda + e_0 = 0,$$

If and only if λ is inside the unit circle, d-RTBP is linear stable.

where

$$e_0 \simeq z^{10} + (5+12\nu-12\nu^2)z^8 + (10+48\nu-48\nu^2)z^6 + (10+48\nu-48\nu^2)z^4 + 5z^2 + 1,$$
$$e_1 \simeq 6z^{10} + (10+24\nu-24\nu^2)z^8 - (4+192\nu-192\nu^2)z^6 - (12+480\nu-480\nu^2)z^4 - 2z^2 + 2,$$
$$e_2 \simeq 15z^{10} - (5+12\nu-12\nu^2)z^8 - (42+48\nu-48\nu^2)z^6 - (10-1680\nu+1680\nu^2)z^4 + 11z^2 - 1,$$
$$e_3 \simeq 20z^{10} - (20+48\nu-48\nu^2)z^8 - (56-384\nu+384\nu^2)z^6 + (24-2496\nu+2496\nu^2)z^4 + 36z^2 - 4,$$
$$z = \tan\frac{\Omega\Delta t}{4}$$

Linear stablity of triangular equilibrium points in d-RTBP

Jury's stability test Characteristic eq. : $\underline{a_6}\lambda^6+\underline{a_5}\lambda^5+\underline{a_4}\lambda^4+\underline{a_3}\lambda^3+\underline{a_2}\lambda^2+\underline{a_1}\lambda+\underline{a_0}=0$

	a_6	a_5	a_4	a_3	a_2	a_1	a_0
$-)$	a_0k_a	a_1k_a	a_2k_a	a_3k_a	a_4k_a	a_5k_a	
	b_0	b_1	b_2	b_3	b_4	b_5	
$-)$	b_5k_b	b_4k_b	b_3k_b	b_2k_b	b_1k_b		
	c_0	c_1	c_2	c_3	c_4		
$-)$	c_4k_c	c_3k_c	c_2k_c	c_1k_c			
	d_0	d_1	d_2	d_3			
$-)$	d_3k_d	d_2k_d	d_1k_d				
	e_0	e_1	e_2				
$-)$	e_2k_e	e_1k_e					
	f_0	f_1					
$-)$	f_1k_f						
	g_0						

$k_a=a_0/a_6$

$k_b=b_5/b_0$

$k_c=c_4/c_0$

$k_d=d_3/d_0$

$k_e=e_2/e_0$

$k_f=f_1/f_0$

Raible's Tabulation

Nonsingular case

All elements of the 1st column $\left(a_6, b_0, c_0, d_0, e_0, f_0 \text{ and } g_0\right)$ are nonzero.

Linear stablity of triangular equilibrium points in d-RTBP

Jury's stability test CE : $\underline{a_6}\lambda^6 + \underline{a_5}\lambda^5 + \underline{a_4}\lambda^4 + \underline{a_3}\lambda^3 + \underline{a_2}\lambda^2 + \underline{a_1}\lambda + \underline{a_0} = 0$

	$\underline{a_6}$	$\underline{a_5}$	$\underline{a_4}$	$\underline{a_3}$	$\underline{a_2}$	$\underline{a_1}$	$\underline{a_0}$
$-)$	a_0k_a	a_1k_a	a_2k_a	a_3k_a	a_4k_a	a_5k_a	
	b_0	b_1	b_2	b_3	b_4	b_5	
$-)$	b_5k_b	b_4k_b	b_3k_b	b_2k_b	b_1k_b		
	c_0	c_1	c_2	c_3	c_4		
$-)$	c_4k_c	c_3k_c	c_2k_c	c_1k_c			
	$\boxed{d_0}$	$\boxed{d_1}$	$\boxed{d_2}$	$\boxed{d_3}$			
$-)$	d_3k_d	d_2k_d	d_1k_d				
	e_0	e_1	e_2				
$-)$	e_2k_e	e_1k_e					
	f_0	f_1					
$-)$	f_1k_f						
	g_0						

$k_a = a_0/a_6$

$k_b = b_5/b_0$

$k_c = c_4/c_0$

$k_d = d_3/d_0$

$k_e = e_2/e_0$

$k_f = f_1/f_0$

Raible's Tabulation

Nonsingular case

All elements of the 1st column $\left(\boxed{a_6}, \boxed{b_0}, \boxed{c_0}, \boxed{d_0}, \boxed{e_0}, \boxed{f_0} \text{ and } \boxed{g_0}\right)$ are nonzero.

and

A complete row of zeros is never present. (cf. $(d_0, d_1, d_2, d_3) \neq (0, 0, 0, 0)$)

Linear stablity of triangular equilibrium points in d-RTBP

Jury's stability test

CE : $\underline{a_6}\lambda^6 + \underline{a_5}\lambda^5 + \underline{a_4}\lambda^4 + \underline{a_3}\lambda^3 + \underline{a_2}\lambda^2 + \underline{a_1}\lambda + \underline{a_0} = 0$

	$\underline{a_6}$	$\underline{a_5}$	$\underline{a_4}$	$\underline{a_3}$	$\underline{a_2}$	$\underline{a_1}$	$\underline{a_0}$
$-)$	$a_0 k_a$	$a_1 k_a$	$a_2 k_a$	$a_3 k_a$	$a_4 k_a$	$a_5 k_a$	
	$\boxed{b_0}$	b_1	b_2	b_3	b_4	b_5	
$-)$	$b_5 k_b$	$b_4 k_b$	$b_3 k_b$	$b_2 k_b$	$b_1 k_b$		
	$\boxed{c_0}$	c_1	c_2	c_3	c_4		
$-)$	$c_4 k_c$	$c_3 k_c$	$c_2 k_c$	$c_1 k_c$			
	$\boxed{d_0}$	d_1	d_2	d_3			
$-)$	$d_3 k_d$	$d_2 k_d$	$d_1 k_d$				
	$\boxed{e_0}$	e_1	e_2				
$-)$	$e_2 k_e$	$e_1 k_e$					
	$\boxed{f_0}$	f_1					
$-)$	$f_1 k_f$						
	$\boxed{g_0}$						

$k_a = a_0/a_6$

$k_b = b_5/b_0$

$k_c = c_4/c_0$

$k_d = d_3/d_0$

$k_e = e_2/e_0$

$k_f = f_1/f_0$

Raible's Tabulation

In the nonsingular case , and $a_6 > 0$,

- Number of postive caluculated elements in the 1st column $\left(\boxed{b_0}, \boxed{c_0}, \cdots, \boxed{g_0}\right)$
  $=$ Number of roots inside the unit circle
- Number of negative caluculated elements in the 1st column $\left(\boxed{b_0}, \boxed{c_0}, \cdots, \boxed{g_0}\right)$
  $=$ Number of roots outside the unit circle

Linear stablity of triangular equilibrium points in d-RTBP

Jury's stability test CE : $\underline{a_6}\lambda^6 + \underline{a_5}\lambda^5 + \underline{a_4}\lambda^4 + \underline{a_3}\lambda^3 + \underline{a_2}\lambda^2 + \underline{a_1}\lambda + \underline{a_0} = 0$

	$\underline{a_6}$	$\underline{a_5}$	$\underline{a_4}$	$\underline{a_3}$	$\underline{a_2}$	$\underline{a_1}$	$\underline{a_0}$
−)	a_0k_a	a_1k_a	a_2k_a	a_3k_a	a_4k_a	a_5k_a	
	$\boxed{b_0}$	b_1	b_2	b_3	b_4	b_5	
−)	b_5k_b	b_4k_b	b_3k_b	b_2k_b	b_1k_b		
	$\boxed{c_0}$	c_1	c_2	c_3	c_4		
−)	c_4k_c	c_3k_c	c_2k_c	c_1k_c			
	$\boxed{d_0}$	d_1	d_2	d_3			
−)	d_3k_d	d_2k_d	d_1k_d				
	$\boxed{e_0}$	e_1	e_2				
−)	e_2k_e	e_1k_e					
	$\boxed{f_0}$	f_1					
−)	f_1k_f						
	$\boxed{g_0}$						

$k_a = a_0/a_6$

$k_b = b_5/b_0$

$k_c = c_4/c_0$

$k_d = d_3/d_0$

$k_e = e_2/e_0$

$k_f = f_1/f_0$

Raible's Tabulation

In the nonsingular case , and $a_6 > 0$, the discrete system is **linear stable**
when **all roots are in the unit circle** , namely, $\boxed{b_0 \geq 0, c_0 \geq 0, d_0 \geq 0, e_0 \geq 0, f_0 \geq 0, g_0 \geq 0}$.

Linear stablity of triangular equilibrium points in d-RTBP

Jury's stability test Singular case

CE : $\tilde{a}_6\lambda^6 + \tilde{a}_5\lambda^5 + \tilde{a}_4\lambda^4 + \tilde{a}_3\lambda^3 + \tilde{a}_2\lambda^2 + \tilde{a}_1\lambda + \tilde{a}_0 = 0$,
where $\tilde{a}_n = a_n(1 + n\varepsilon)$.

	$\tilde{a}_6$	$\tilde{a}_5$	$\tilde{a}_4$	$\tilde{a}_3$	$\tilde{a}_2$	$\tilde{a}_1$	$\tilde{a}_0$
$-)$	$\tilde{a}_0\tilde{k}_a$	$\tilde{a}_1\tilde{k}_a$	$\tilde{a}_2\tilde{k}_a$	$\tilde{a}_3\tilde{k}_a$	$\tilde{a}_4\tilde{k}_a$	$\tilde{a}_5\tilde{k}_a$	
	$\tilde{b}_0$	$\tilde{b}_1$	$\tilde{b}_2$	$\tilde{b}_3$	$\tilde{b}_4$	$\tilde{b}_5$	
$-)$	$\tilde{b}_5\tilde{k}_b$	$\tilde{b}_4\tilde{k}_b$	$\tilde{b}_3\tilde{k}_b$	$\tilde{b}_2\tilde{k}_b$	$\tilde{b}_1\tilde{k}_b$		
	$\tilde{c}_0$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{c}_3$	$\tilde{c}_4$		
$-)$	$\tilde{c}_4\tilde{k}_c$	$\tilde{c}_3\tilde{k}_c$	$\tilde{c}_2\tilde{k}_c$	$\tilde{c}_1\tilde{k}_c$			
	$\tilde{d}_0$	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$			
$-)$	$\tilde{d}_3\tilde{k}_d$	$\tilde{d}_2\tilde{k}_d$	$\tilde{d}_1\tilde{k}_d$				
	$\tilde{e}_0$	$\tilde{e}_1$	$\tilde{e}_2$				
$-)$	$\tilde{e}_2\tilde{k}_e$	$\tilde{e}_1\tilde{k}_e$					
	$\tilde{f}_0$	$\tilde{f}_1$					
$-)$	$\tilde{f}_1\tilde{k}_f$						
	$\tilde{g}_0$						

$$\tilde{k}_a = \tilde{a}_0/\tilde{a}_6$$
$$\tilde{k}_b = \tilde{b}_5/\tilde{b}_0$$
$$\tilde{k}_c = \tilde{c}_4/\tilde{c}_0$$
$$\tilde{k}_d = \tilde{d}_3/\tilde{d}_0$$
$$\tilde{k}_e = \tilde{e}_2/\tilde{e}_0$$
$$\tilde{k}_f = \tilde{f}_1/\tilde{f}_0$$

Raible's Tabulation

In the case of $\varepsilon > 0$, $\tilde{b}_0, \tilde{c}_0, \cdots, \tilde{g}_0$ are all positive.
and
In the case of $\varepsilon < 0$, $\tilde{b}_0, \tilde{c}_0, \cdots, \tilde{g}_0$ are all negative.

$\Longrightarrow$ All roots of CE are on the unit circle. Namely, the dicrete system is **linear stable** (but is not asymptotic stable).

Linear stablity of triangular equilibrium points in d-RTBP

From **Jury's stability test**

$$\begin{aligned} 0 \leq \nu &= \frac{m_2}{M} < \frac{1}{2} - \frac{\sqrt{69}}{18} + O(z^2) \\ &= \boxed{\frac{1}{2} - \frac{\sqrt{69}}{18}} + O((\Delta t)^2) \\ &= \boxed{\text{Upper limit of linear stability condition for RTBP}} + O((\Delta t)^2). \end{aligned}$$

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Conclusion : The linear stability conditions of the triangular Lagrangian equilibrium points in the d-RTBP are accordance with those in the original RTBP up to order one of Δt.