Celestial Mechanics and N-body Dynamics @Osaka

1 Sep 2011

General Relativistic Three-body Problem

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(Hirosaki U, Japan)

--- My talk about Hirosaki papers ---

Chiba, Imai, HA,
Mon. Not. Roy. Astr. S, 377, 269 (2007)
Arxiv:astro-ph/0609773.

Imai, Chiba, HA,
Phys. Rev. Lett. 98, 201102 (2007)
Arxiv:gr-qc/0702076.

Torigoe, Hattori, HA,
Phys. Rev. Lett. 102, 251101 (2009)
Arxiv:gr-qc/0906.1448

HA,
Phys. Rev. D 80, 064021 (2009)
Arxiv:gr-qc/1010.2284

Yamda, HA,
Phys. Rev. D 82, 104019 (2010)
Arxiv:gr-qc/1010.2284

Yamda, HA,
Phys. Rev. D 83, 024040 (2011)
Arxiv:gr-qc/1011.2007

Ichita, Yamda, HA,
Phys. Rev. D 83, 084026 (2011)
Arxiv:gr-qc/1011.3886

See Yamada poster

N-body Problem in Newton gravity

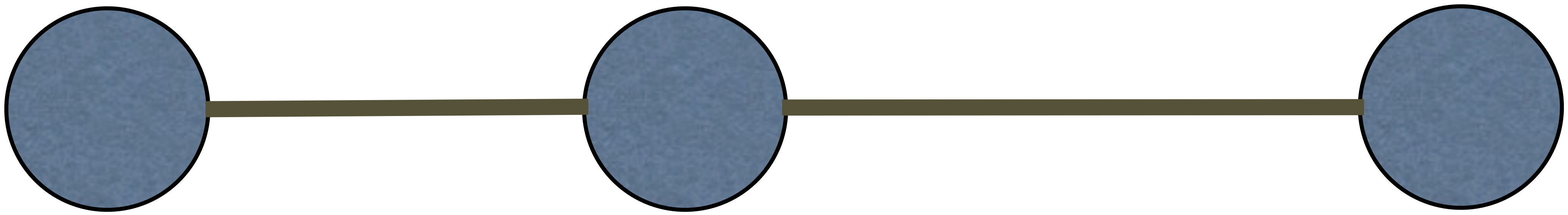
2-body problem

solved by (E, L)

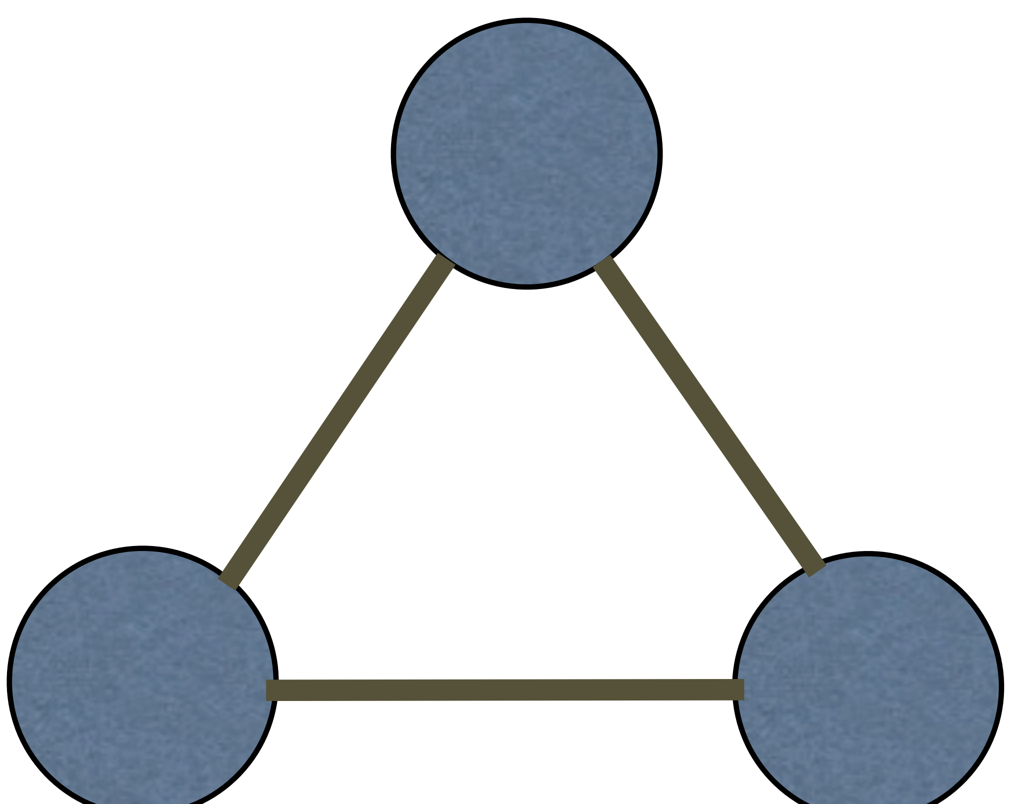
elliptic	$E < 0$
parabolic	$E = 0$
hyperbolic	$E > 0$

3-body

Euler's collinear solution (1765)



Lagrange's triangle (1772)



Poincare

N = 3 (or more)

impossible to describe
all the solutions
to the N-body problem.

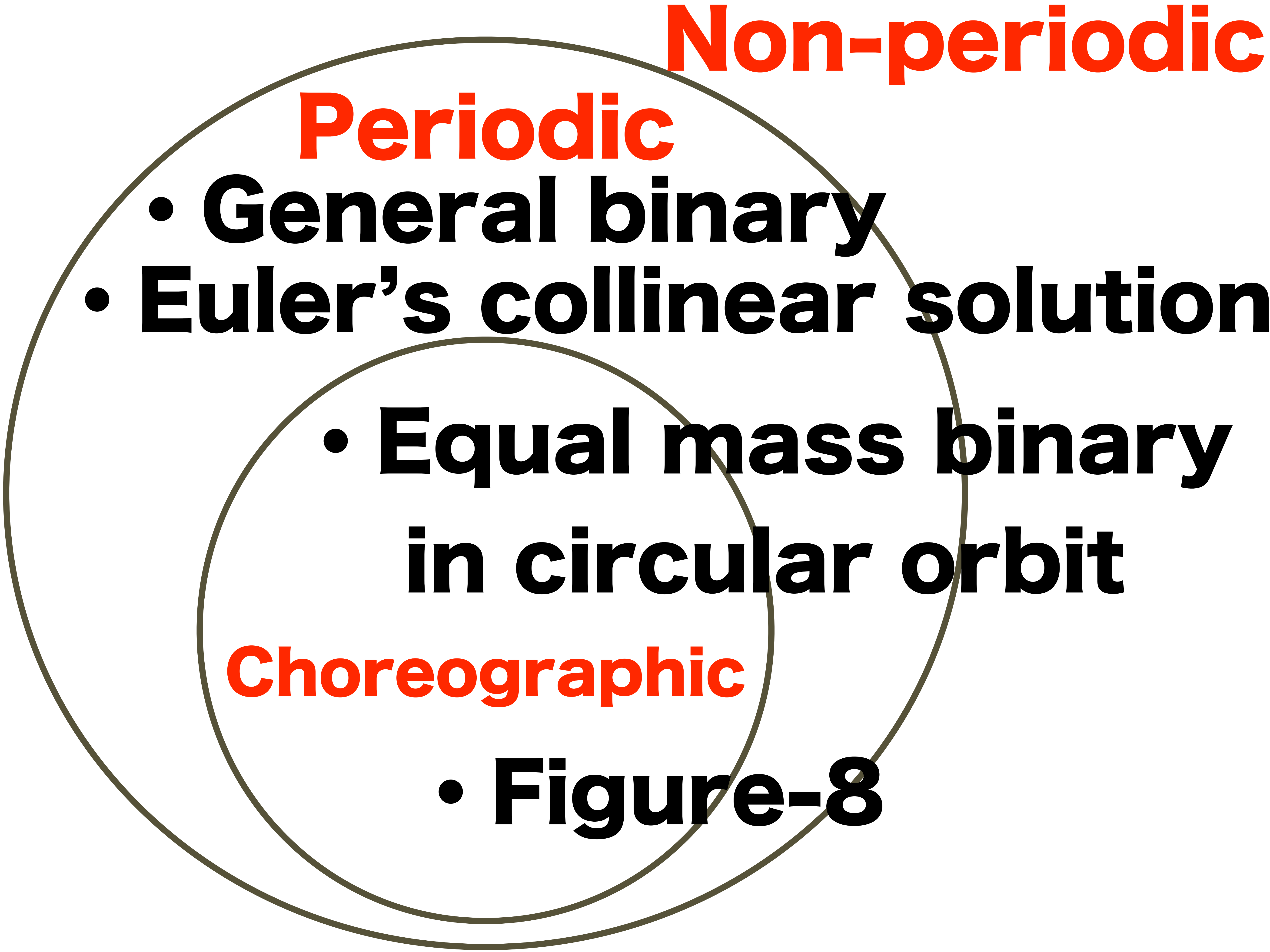
# of new solutions
is increasing.

Remarkable one
was found

Figure-eight solution!

Moore,
Phys. Rev. Lett. 70, 3675 (1993)

Chenciner, Montgomery,
Ann. Math. 152, 881 (2000)


Non-periodic
Periodic
• General binary
• Euler's collinear solution
• Equal mass binary
in circular orbit
Choreographic
• Figure-8

Let us re-examine

3-body problem

in the framework of

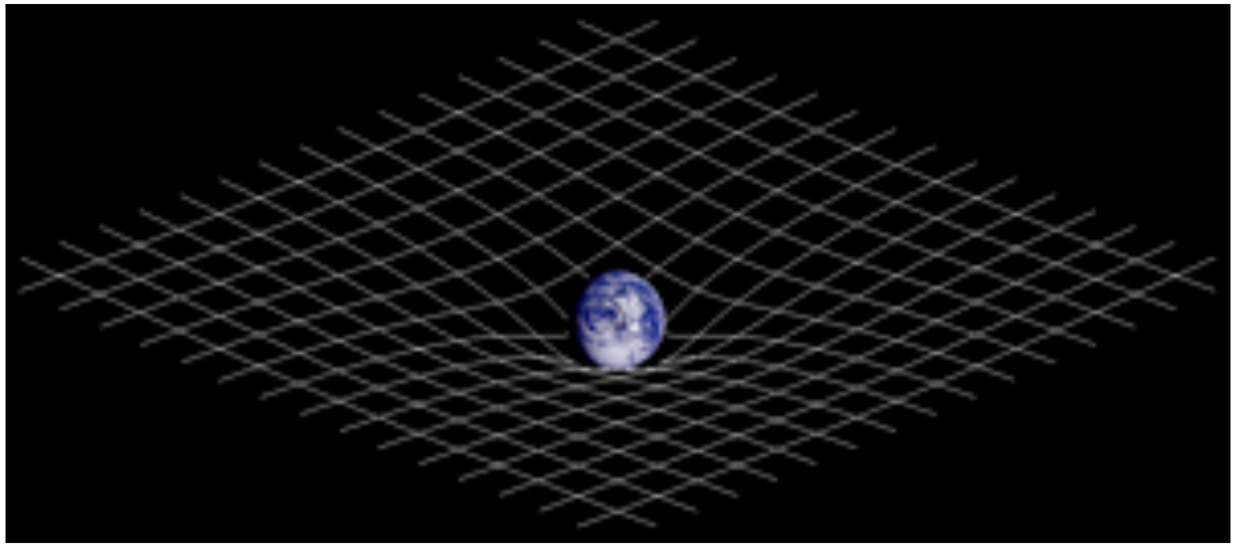
general relativity

GR = General Relativity

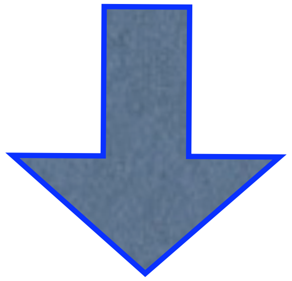
Newton

Gravity = Force

Einstein

Gravity = 

Curved Space-time



light ray bends

gravitational waves

$$G_{\mu\nu} = T_{\mu\nu}$$

Space-time Geometry

Matter Energy -Momentum

Post-Newtonian approx.

Newton + 1PN + 2PN + ...

$$\left(\frac{v}{c}\right)^2 \qquad \left(\frac{v}{c}\right)^4$$

Dominant corrections

General relativistic effects

Periastron advance

Mercury

Time delay

GPS

Light bending

Gravitational Lens

Binary pulser

Hulse-Taylor

GW=Gravitational Waves

Tiny ripples of

a curved space-time

Generated by

accelerated masses

No direct detection so far

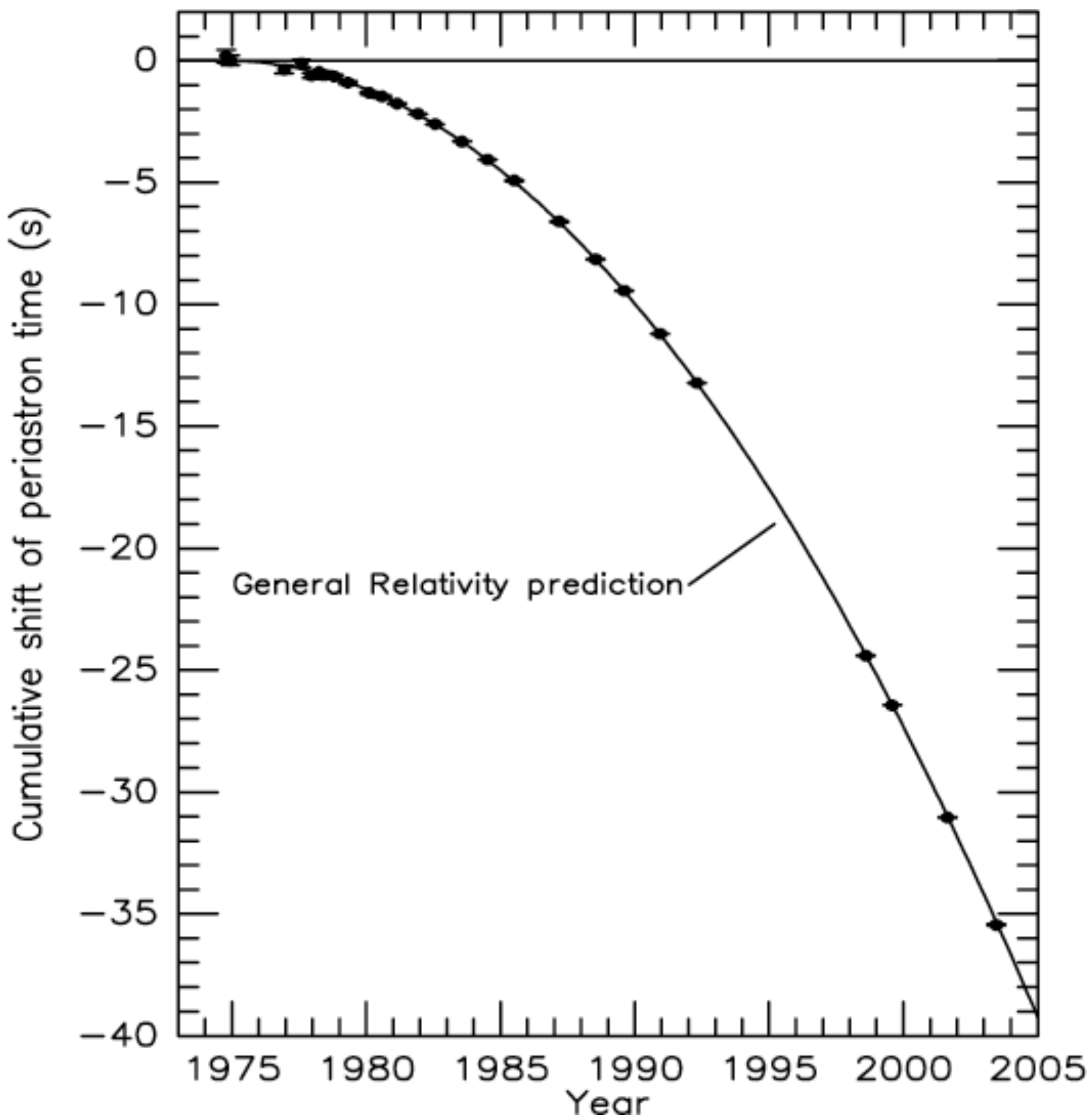
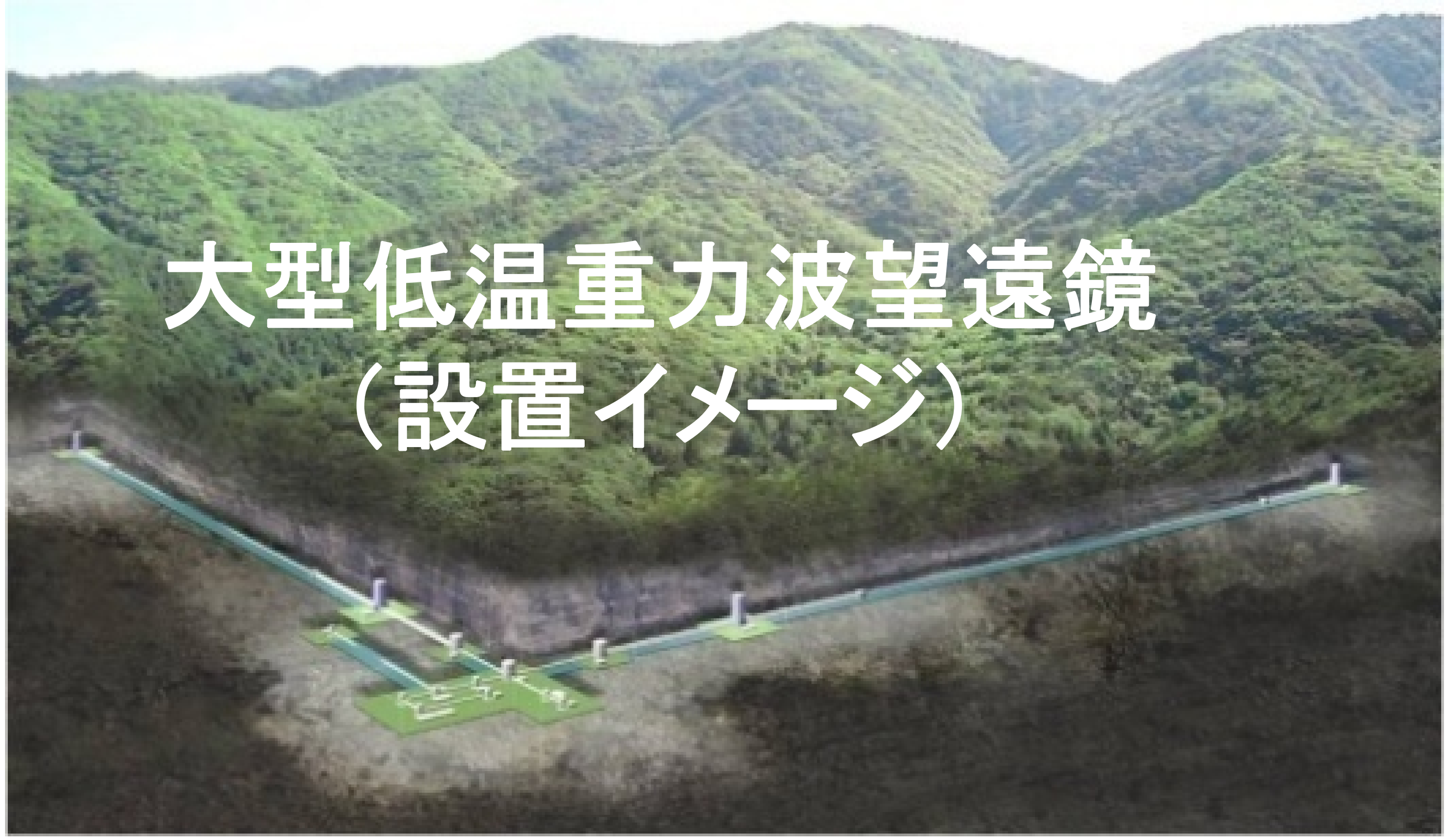


Figure 7: Plot of the cumulative shift of the periastron time from 1975 – 2005. The points are data, the curve is the GR prediction. The gap during the middle 1990s was caused by a closure of Arecibo for upgrading [272].

Will, LRR (06)

indirect evidence

by Binary Pulser



LCGT->KAGRA(Japan)

LIGO(US)

Part 1: Choreography

Part 2: Euler+Lagrange's solutions

In Celestial Mechanics,

a solution is

‘choreographic’

if

every massive particles
move periodically
in a single closed orbit

1)

Implication of Choreography to GR

2)

Effects of GR to Choreography

1) Implication of Choreography to GR

2) Effects of GR to Choreography

Promising GW sources

N=1

Rapidly Rotating Star

N=2

Compact Binary System

N=3 (or more)

much less attention

Because of Chaos

irregular waveform

↓

difficult to detect

Our question

Can three (or more) bodies generate period GW?

Ans.

Yes!

Chiba, Imai, HA,
Mon. Not. Roy. Astr. S, 377, 269 (2007)
Arxiv:astro-ph/0609773.

One example

Figure-8

Assumptions

The same plane

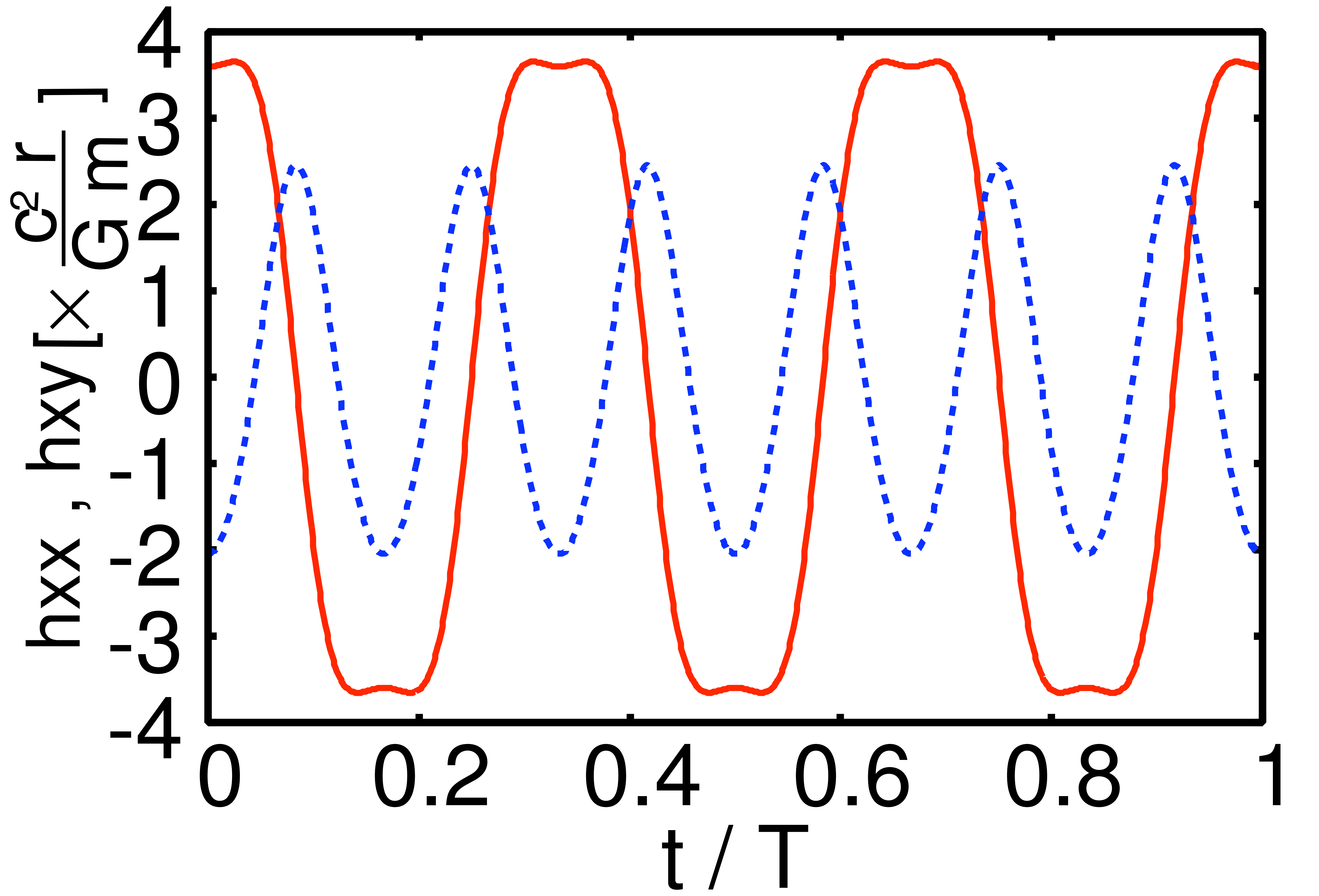
The same mass

Computing Waveform
via Quadrupole formula

$$h_{ij}^{TT} = \frac{2G\ddot{Q}_{ij}}{rc^4} + O\left(\frac{1}{r^2}\right)$$

$$Q_{ij} = I_{ij} - \delta_{ij}\frac{I_{kk}}{3}$$

$$I_{ij} = \sum_{A=1}^{N} m_A x_A^i x_A^j$$



4
3
2
1
0
-1
-2
-3
-4
0
0.2
0.4
0.6
0.8
1
t / T
hxx , hxy [× c² r / G m]

1) Implication of Choreography to GR

2) Effects of GR to Choreography

2nd question

Newton’s EOM is OK?

Ans.

No!

Imai, Chiba, HA,
Phys. Rev. Lett. 98, 201102 (2007)
Arxiv:gr-qc/0702076.

Einstein-Infeld-Hoffman Equation of motion

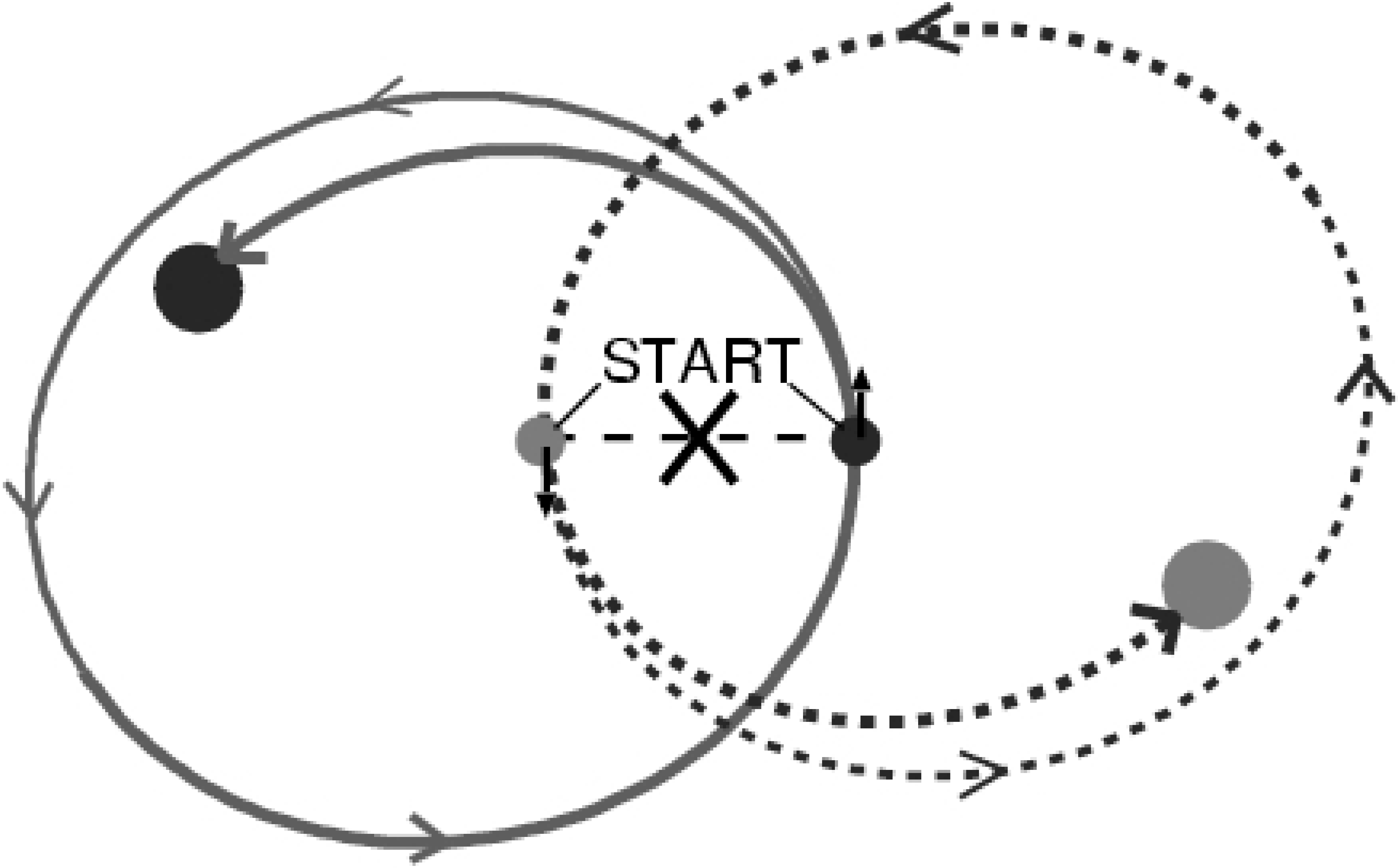
G = c = 1

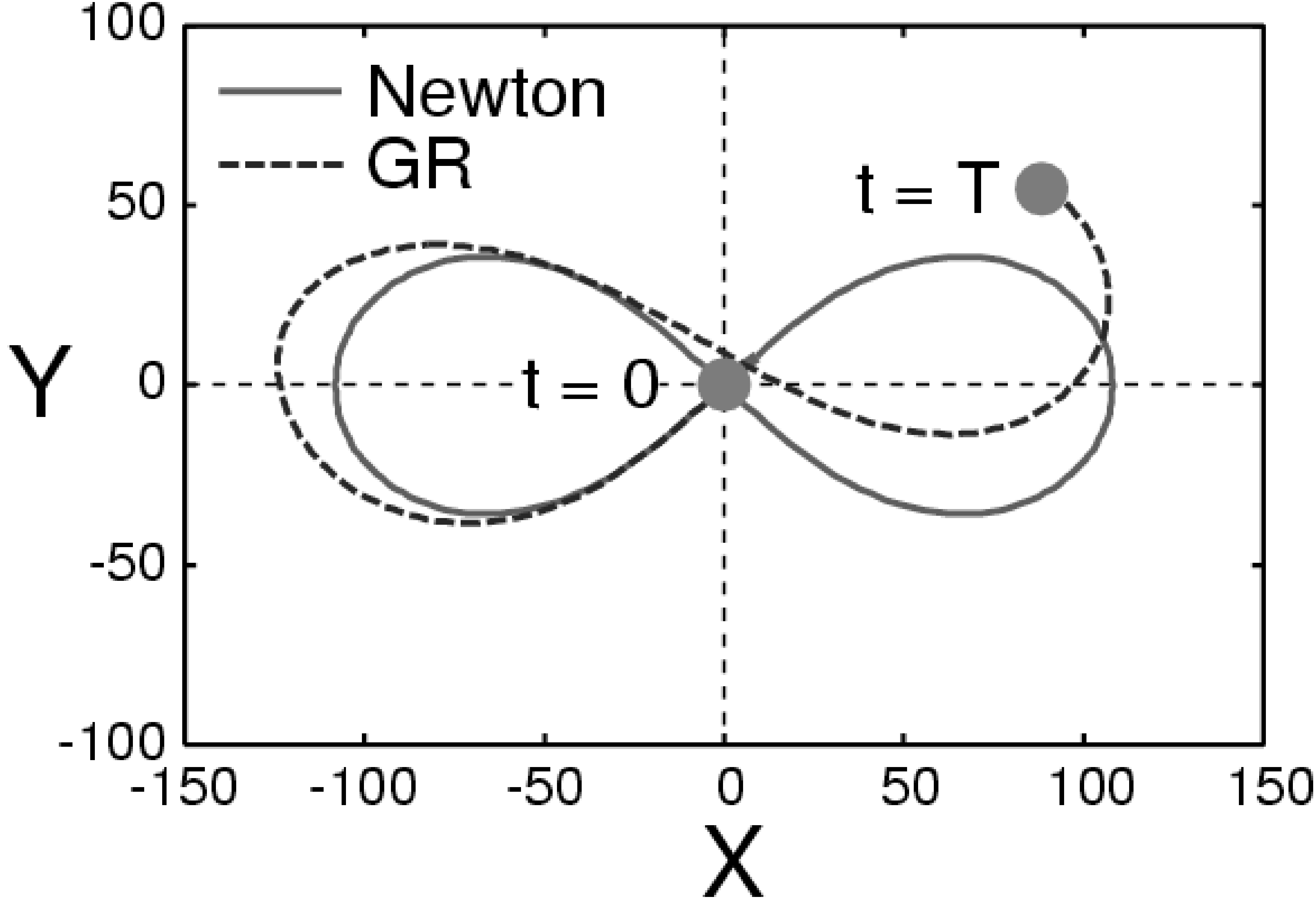
$$
\begin{aligned}
\frac{d^2\boldsymbol{x}_K}{dt^2} = \sum_{A\neq K} \boldsymbol{r}_{AK}\frac{m_A}{r_{AK}^3}\Bigg[1 &- 4\sum_{B\neq K}\frac{m_B}{r_{BK}} \\
&- \sum_{C\neq A}\frac{m_C}{r_{CA}}\left(1-\frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right) \\
&+ v_K^2 + 2v_A^2 - 4v_A\cdot v_K - \frac{3}{2}\left(\frac{\boldsymbol{v}_A\cdot\boldsymbol{r}_{AK}}{r_{AK}}\right)^2\Bigg] \\
- \sum_{A\neq K}(\boldsymbol{v}_A-\boldsymbol{v}_K)&\frac{m_A\boldsymbol{r}_{AK}\cdot(3\boldsymbol{v}_A-4\boldsymbol{v}_K)}{r_{AK}^3} \\
+ \frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}&\frac{m_A m_C}{r_{AK}r_{CA}^3}
\end{aligned}
$$

A specific question

For 2 bodies,
orbits **cannot be closed**
because of
periastron advance.

What happens for figure-8 ?


START



Imai, Chiba, HA (2007)

Parametrise initial velocity

$$\vec{v}_1 = k\vec{V} + \xi\frac{m}{\ell^3}(\vec{V}\cdot\vec{\ell})\vec{\ell}$$

$$\vec{v}_2 = k\vec{V} + \xi\frac{m}{\ell^3}(\vec{V}\cdot\vec{\ell})\vec{\ell}$$

$$\vec{v}_3 = \vec{V}$$

$$k = -\frac{1}{2} + \alpha|\vec{V}|^2 + \beta\frac{m}{\ell}$$

$$\vec{P}_{tot} = \vec{L}_{tot} = 0$$

$$\Downarrow$$

$$\alpha = -\frac{3}{16}$$

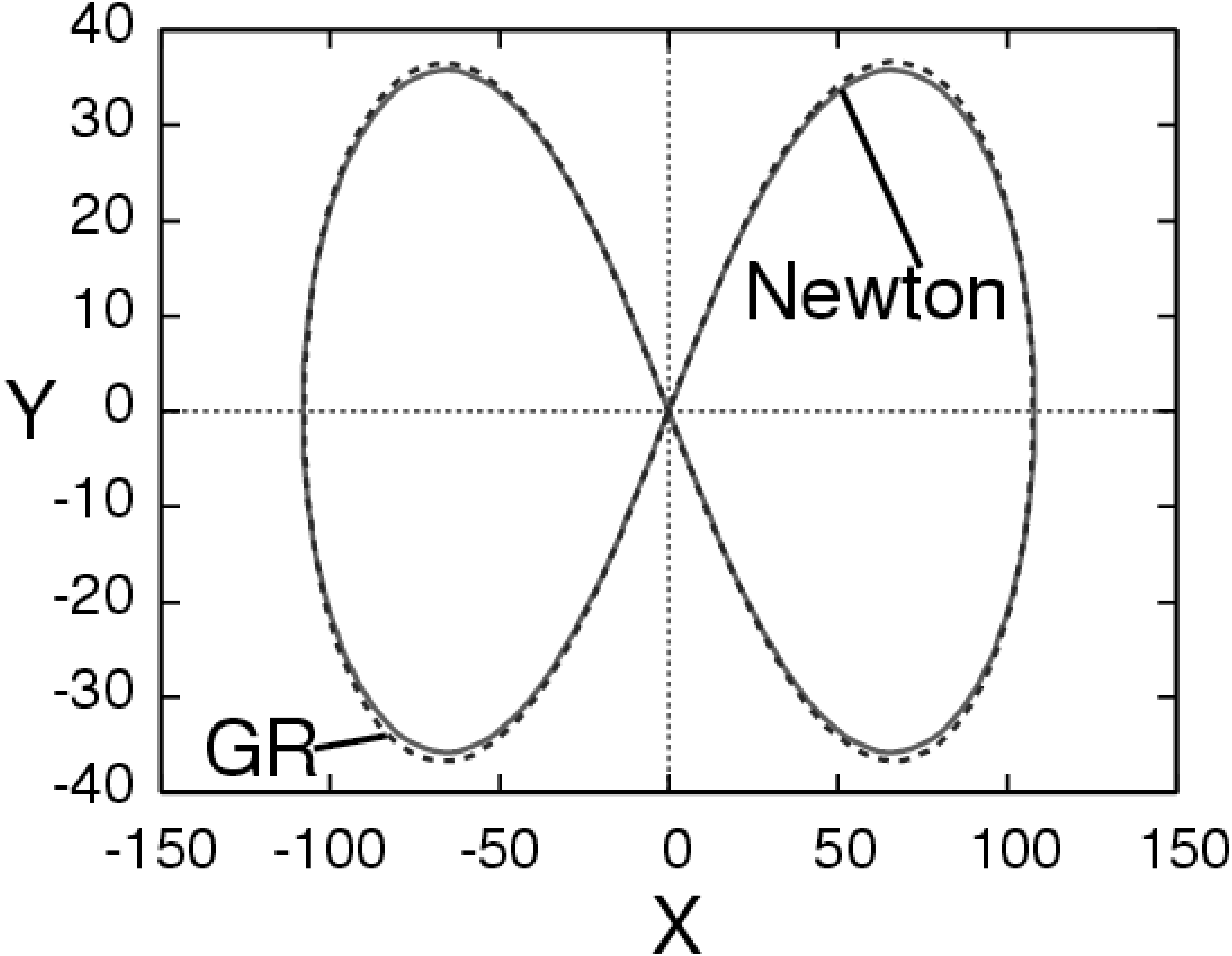
$$\beta = \xi = \frac{1}{8}$$

Remaining degrees of freedom

$$\vec{V} = (V_x, V_y)$$

are numerically determined.

(same as Newton figure-8)



Imai, Chiba, HA (2007)

An extension to 2PN $\left(\frac{v}{c}\right)^4$

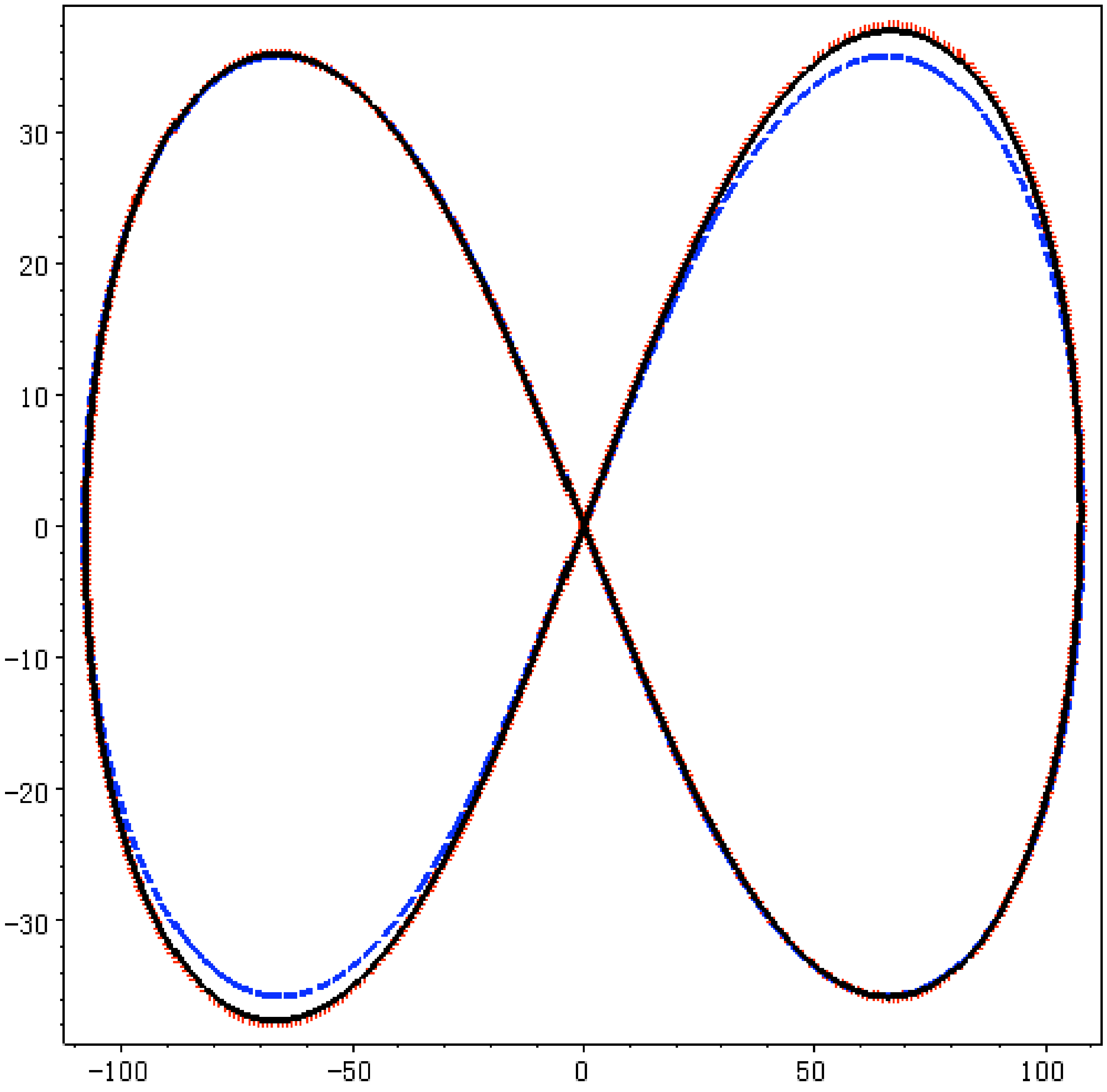
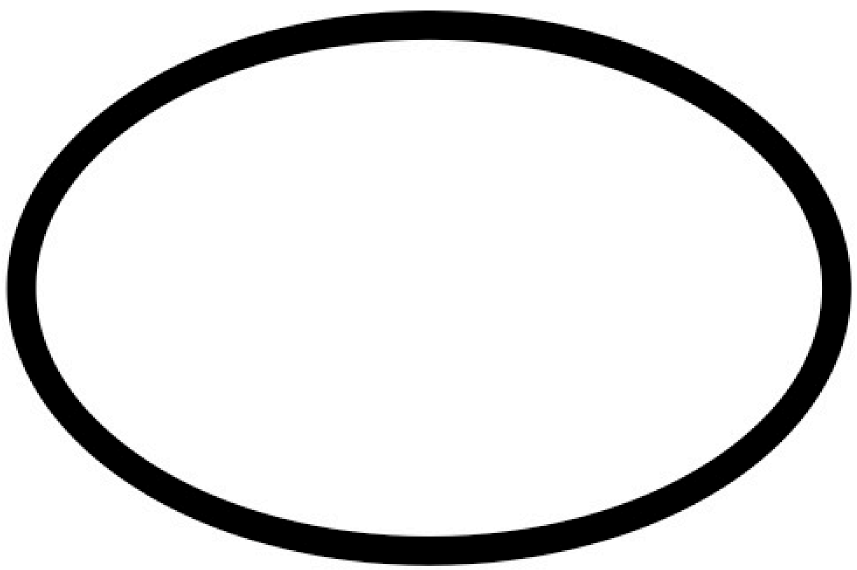
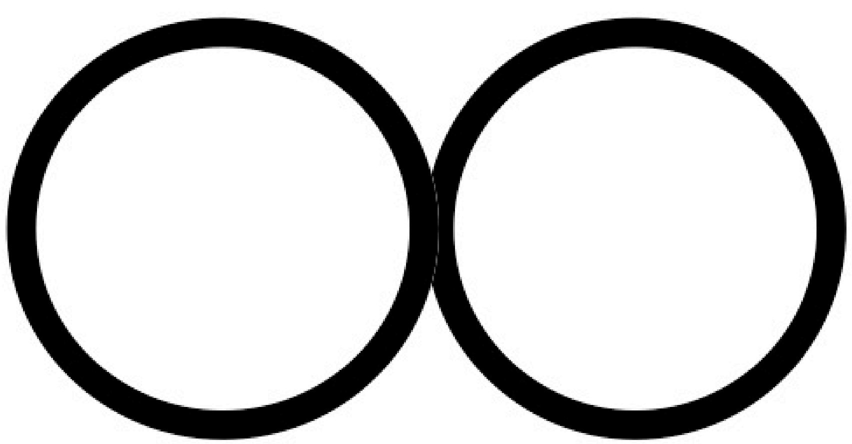


FIG. 9: Comparison of figure-eight motions for $\lambda = 1$. The solid, dotted and dashed lines show the 2PN, 1PN and Newtonian results, respectively.

Lousto, Nakano, Class. Q. Grav. 25, 195019 (2008)

Choreography or Not

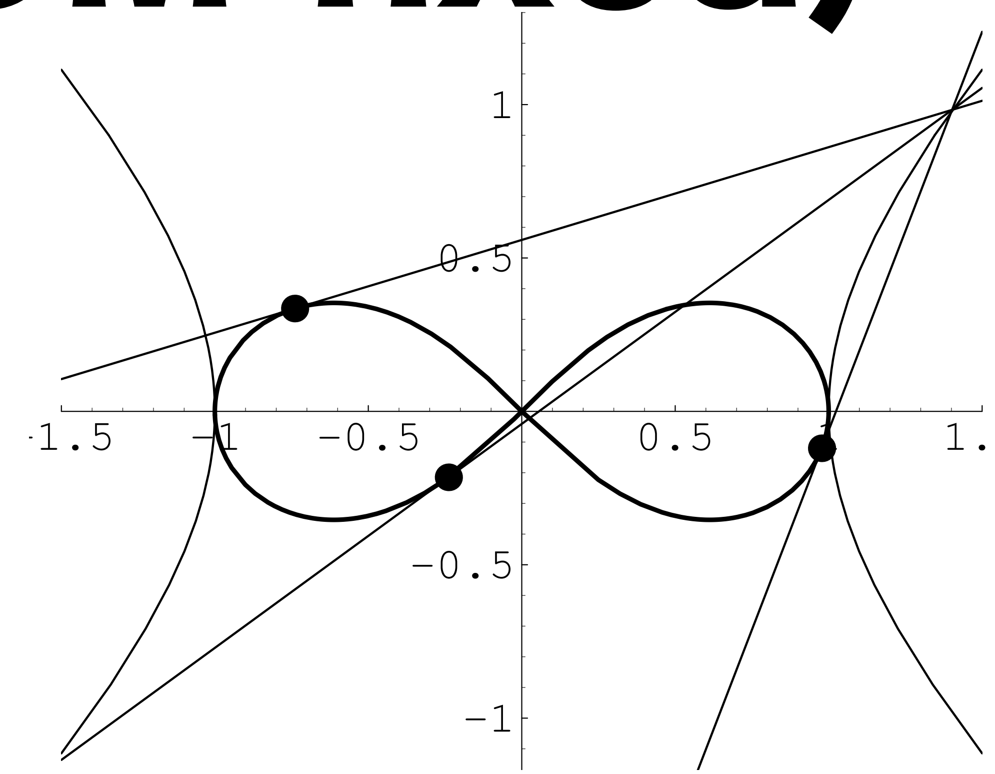
Orbit	Newton	Einstein
	○	× Periastron Shift
	○	○

Fujiwara, Fukuda, Ozaki (2003)

Coplanar 3-body Problem

If total P = 0 (COM fixed)

total L = 0



Tangent lines from 3 bodies always meet at a point

GR figure-8 satisfies 3-tangent line theorem

Because...

In GR, p and v are not always parallel

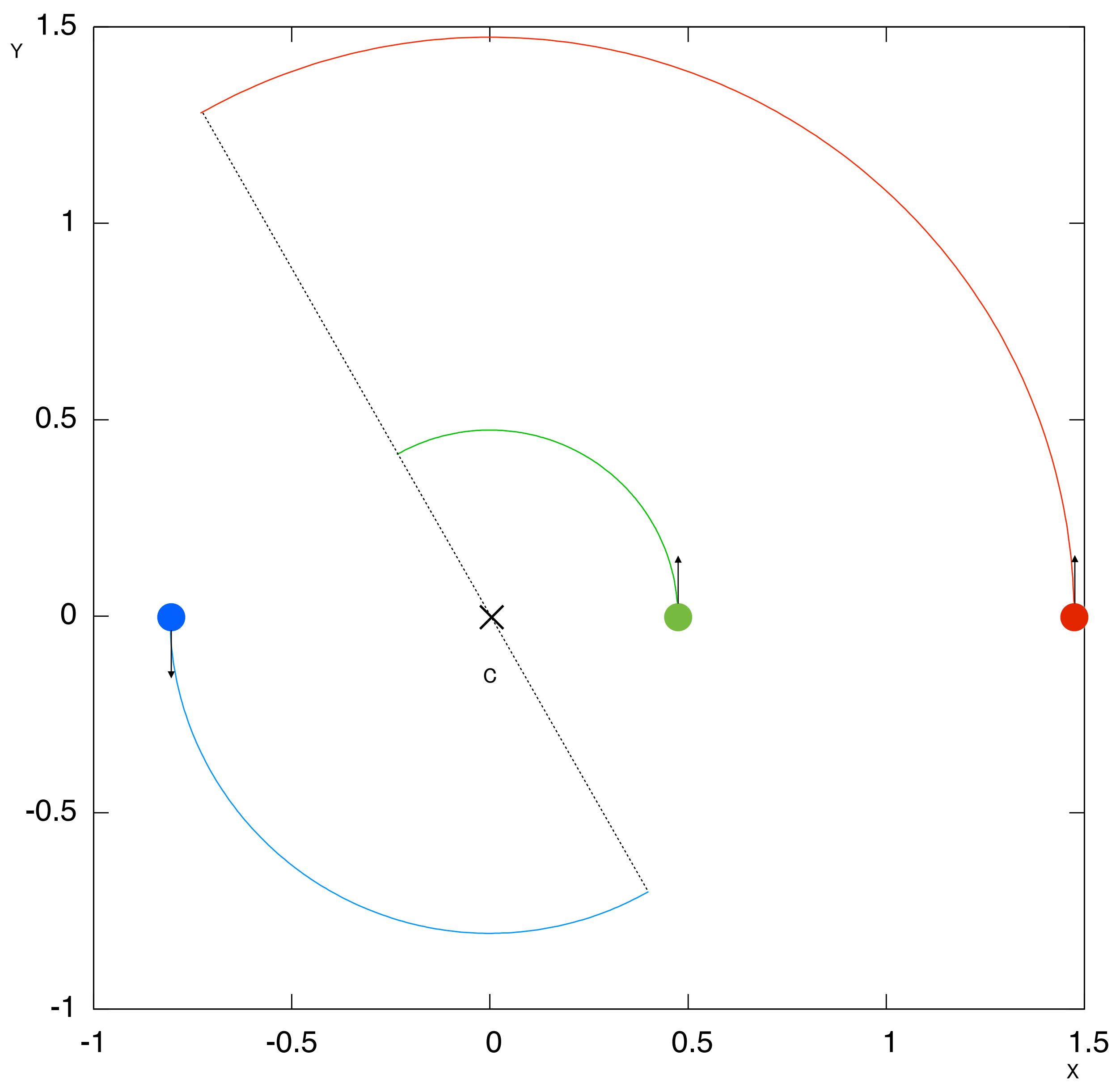
In GR figure-8, p and v are parallel

Part 1: Choreography

Part 2: Euler+Lagrange's solutions

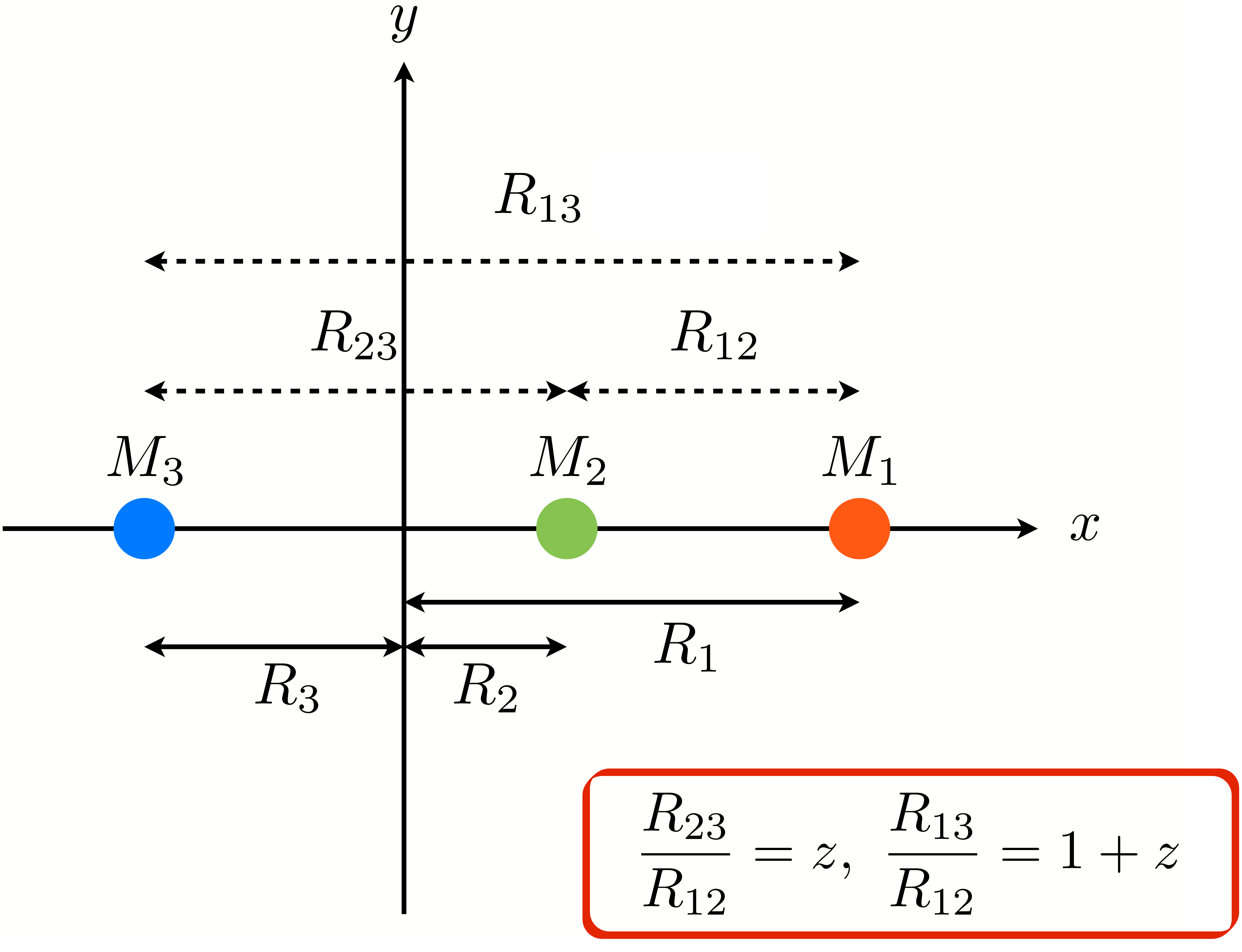
See also Poster by Yamada

GR collinear solution



Euler

Three masses always line up



y
x
R13
R23
R12
M3
M2
M1
R1
R3
R2
R23/R12 = z, R13/R12 = 1 + z

Nonlinear gravity

$$\frac{d^2\boldsymbol{r}_K}{dt^2} = \sum_{A\neq K}\boldsymbol{r}_{AK}\frac{Gm_A}{r_{AK}^3}\left[1-4\sum_{B\neq K}\frac{Gm_B}{c^2r_{BK}}-\sum_{C\neq A}\frac{Gm_C}{c^2r_{CA}}\left(1-\frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)\right.$$
$$\left.+\left(\frac{\boldsymbol{v}_K}{c}\right)^2+2\left(\frac{\boldsymbol{v}_A}{c}\right)^2-4\left(\frac{\boldsymbol{v}_A}{c}\right)\cdot\left(\frac{\boldsymbol{v}_K}{c}\right)-\frac{3}{2}\left(\frac{\left(\frac{\boldsymbol{v}_A}{c}\right)\cdot\boldsymbol{r}_{AK}}{r_{AK}}\right)^2\right]$$
$$-\sum_{A\neq K}\left[\left(\frac{\boldsymbol{v}_A}{c}\right)-\left(\frac{\boldsymbol{v}_K}{c}\right)\right]\frac{Gm_A\boldsymbol{r}_{AK}\cdot\left[3\left(\frac{\boldsymbol{v}_A}{c}\right)-4\left(\frac{\boldsymbol{v}_K}{c}\right)\right]}{r_{AK}^3}$$
$$+\frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{Gm_C}{r_{CA}^3}\frac{Gm_A}{c^2r_{AK}}$$

Correction by velocity

Triple coupling

M1 × M2 × M3

not exist in Newton

Assume

line up

circular motion

Is EIH-EOM satisfied?

Yamada, HA (2010)

$$F(z) \equiv \sum_{k=0}^{7} A_k z^k = 0$$

7th order

$$A_7 = \frac{M}{a}\left[-4 - 2(\nu_1 - 4\nu_3) + 2(\nu_1^2 + 2\nu_1\nu_3 - 2\nu_3^2) - 2\nu_1\nu_3(\nu_1 + \nu_3)\right],$$

$$A_6 = 1 - \nu_3 + \frac{M}{a}\left[-13 - (10\nu_1 - 17\nu_3) + 2(2\nu_1^2 + 8\nu_1\nu_3 - \nu_3^2) + 2(\nu_1^3 - 2\nu_1^2\nu_3 - 3\nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_5 = 2 + \nu_1 - 2\nu_3 + \frac{M}{a}\left[-15 - (18\nu_1 - 5\nu_3) + 4(5\nu_1\nu_3 + 4\nu_3^2) + 6(\nu_1^3 - \nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_4 = 1 + 2\nu_1 - \nu_3 + \frac{M}{a}\left[-6 - 2(5\nu_1 + 2\nu_3) - 4(2\nu_1^2 - \nu_1\nu_3 - 4\nu_3^2) + 2(3\nu_1^3 + \nu_1^2\nu_3 - 2\nu_1\nu_3^2 - 3\nu_3^3)\right],$$

$$A_3 = -(1 - \nu_1 + 2\nu_3) + \frac{M}{a}\left[6 + 2(2\nu_1 + 5\nu_3) - 4(4\nu_1^2 + \nu_1\nu_3 - 2\nu_3^2) + 2(3\nu_1^3 + 2\nu_1^2\nu_3 - \nu_1\nu_3^2 - 3\nu_3^3)\right],$$

$$A_2 = -(2 - 2\nu_1 + \nu_3) + \frac{M}{a}\left[15 - (5\nu_1 - 18\nu_3) - 4(4\nu_1^2 + 5\nu_1\nu_3) + 6(\nu_1^3 + \nu_1^2\nu_3 - \nu_3^3)\right],$$

$$A_1 = -(1 - \nu_1) + \frac{M}{a}\left[13 - (17\nu_1 - 10\nu_3) + 2(\nu_1^2 - 8\nu_1\nu_3 - 2\nu_3^2) + 2(\nu_1^3 + 3\nu_1^2\nu_3 + 2\nu_1\nu_3^2 - \nu_3^3)\right],$$

$$A_0 = \frac{M}{a}\left[4 - 2(4\nu_1 - \nu_3) + 2(2\nu_1^2 - 2\nu_1\nu_3 - \nu_3^2) + 2\nu_1\nu_3(\nu_1 + \nu_3)\right].$$

5th order in Newton Gravity

Yamada, HA (2011)

Descartes rule of signs

and

Slow Motion (PN)

↓ 3 - 2 = 1

Uniqueness

(z = positive)

For the same mass and full length, one can show

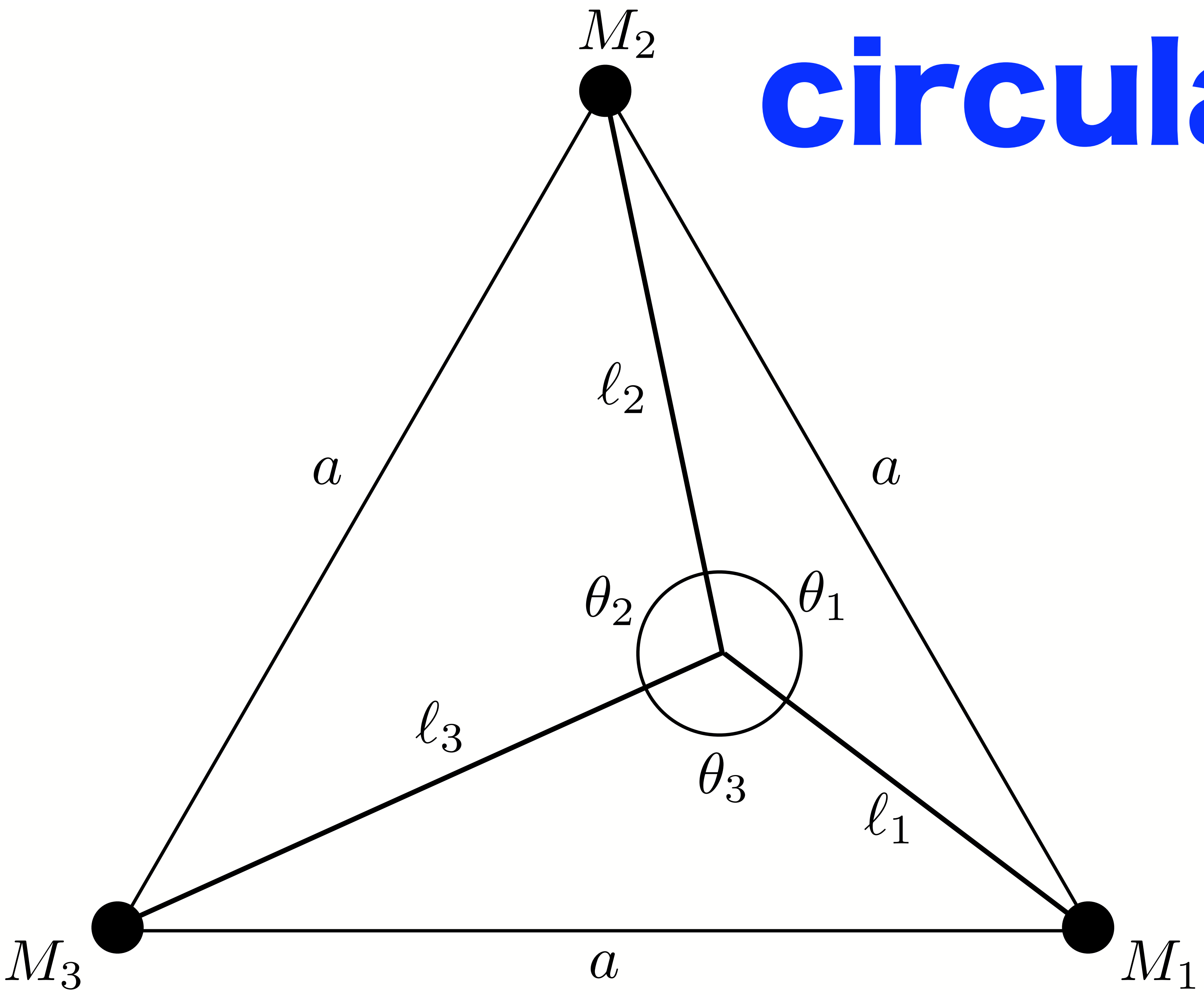
$$\omega < \omega_N$$

GR angular velocity is always smaller

Assume・・

equilateral triangle

circular motion



Equilateral triangular sol.

is possible

in Newton gravity

for three general masses

Ichita, Yamada, HA (2011)

Equilateral triangular sol.

is possible at 1PN in GR

if and only if either

1) Equal finite masses

2) Two equal finite,

one test masses

3) One finite,

two test masses

A little more…

EOM of M1 becomes

$$
\begin{aligned}
-\omega^2 \boldsymbol{x}_1 \;=\; & -\frac{M}{a^3}\boldsymbol{x}_1 + g_{PN1}\boldsymbol{x}_1 \\
& + \frac{\sqrt{3}M}{16a^3}\boldsymbol{n}_{\perp 1}\frac{M_2 M_3 (M_2 - M_3)}{M_2^2 + M_2 M_3 + M_3^2} \\
& \times \left[10 + \frac{a^3}{M^2}\big(-4M_1 + 5M_2 + 5M_3\big)\omega^2\right]
\end{aligned}
$$

M2=M3,

unless test mass

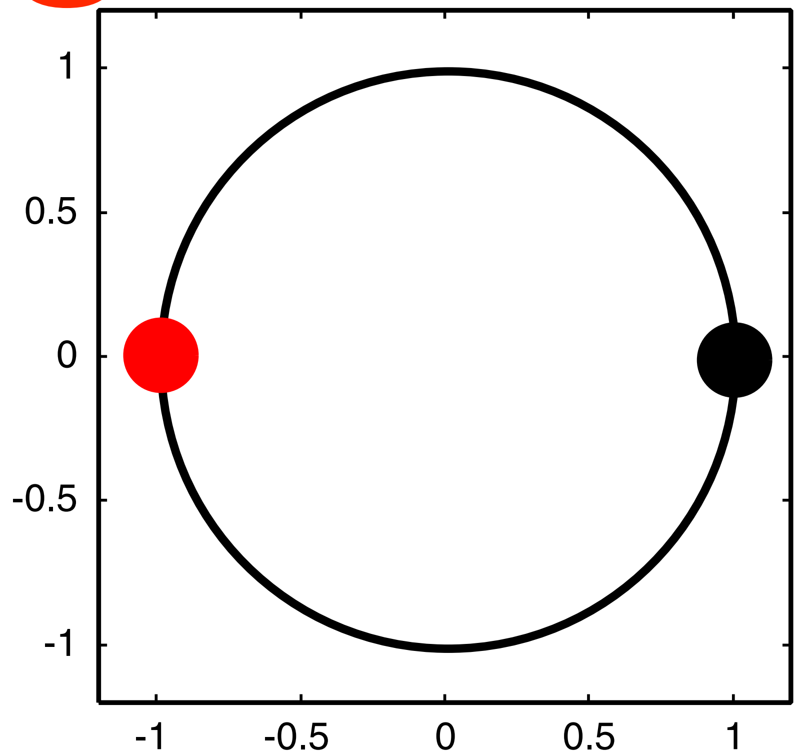
For the same mass and side length, one can show

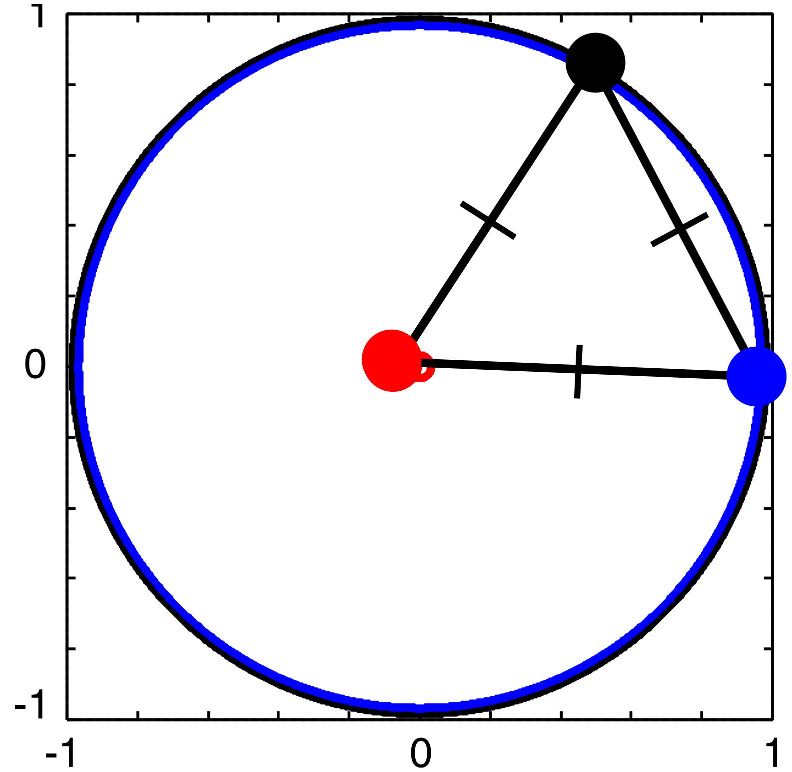
$$\omega < \omega_N$$

GR angular velocity is always smaller

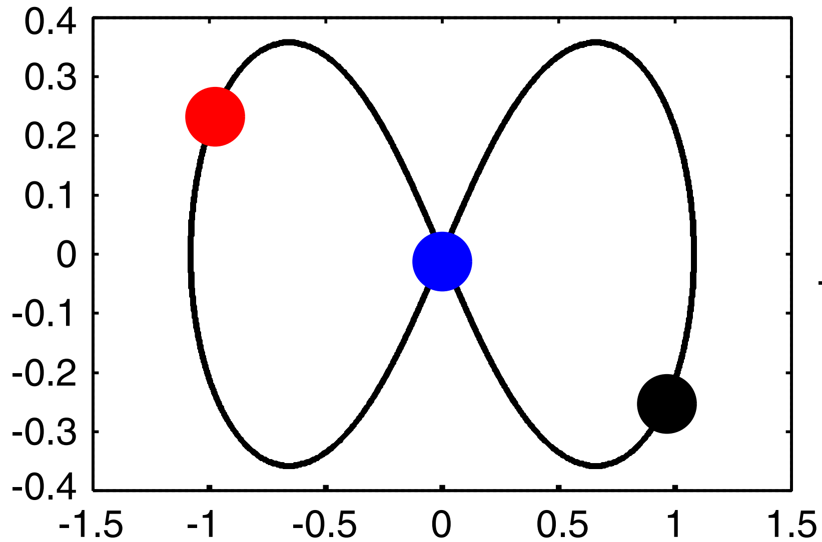
GWs

Torigoe et al. PRL (2009)

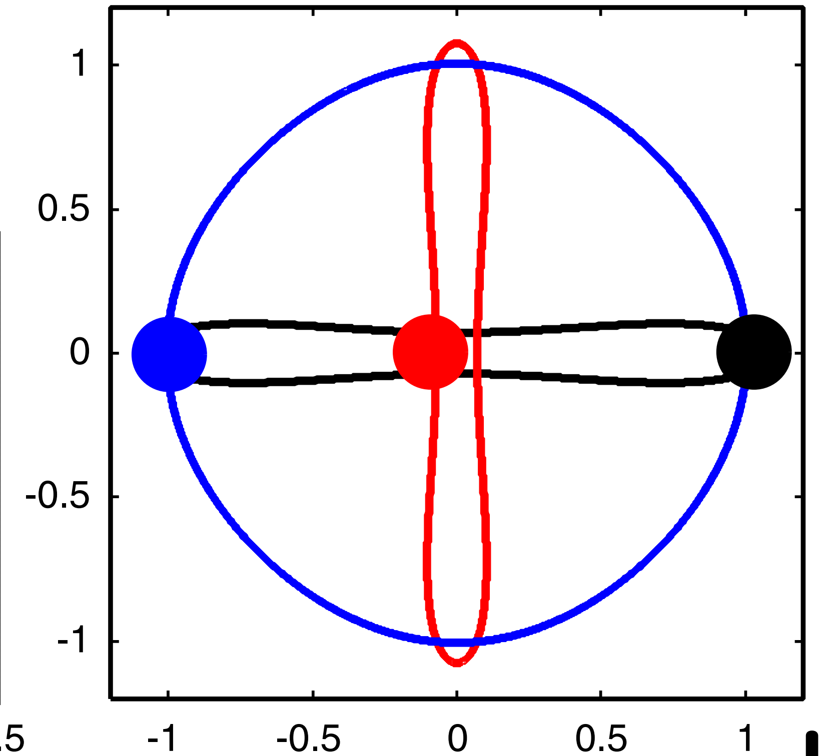




Lagrange's Triangle

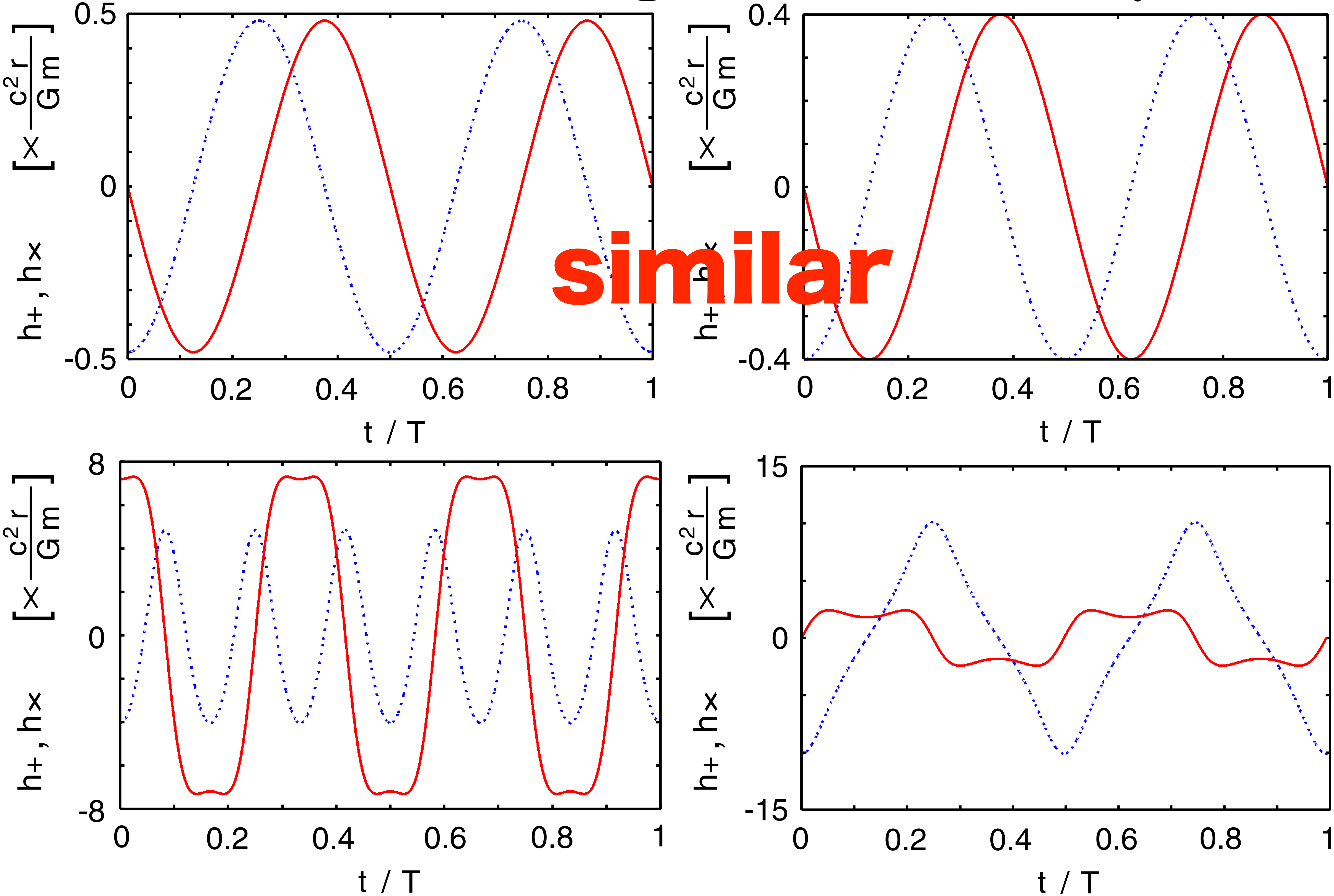


Moore's Figure-8



Henon's Criss-cross

Torigoe et al. PRL (2009)



Obs. z-direction

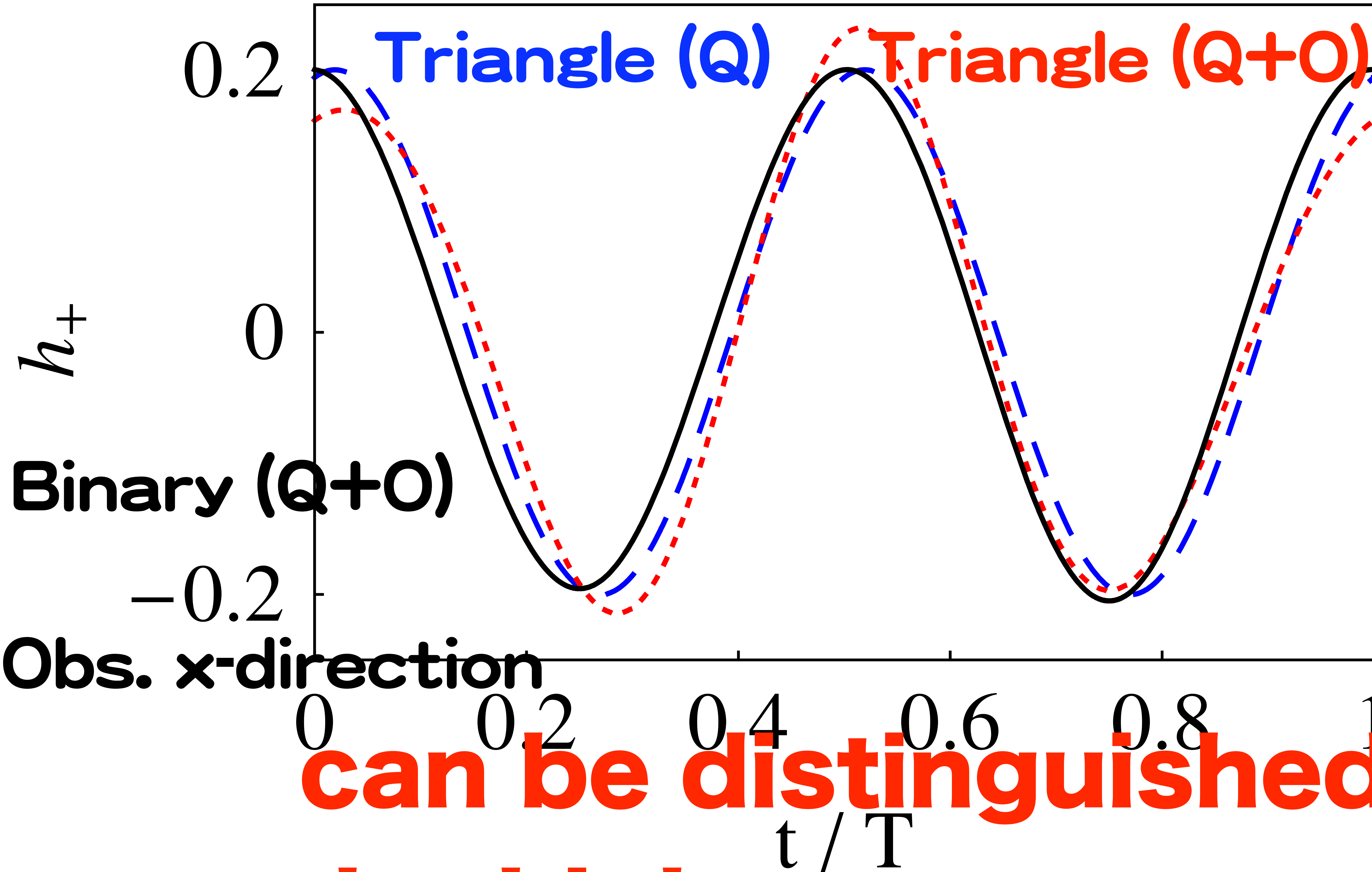
orbital shrinking rate

$$\frac{1}{a}\frac{da}{dt} = -\frac{64}{5}\frac{m_{\rm tot}^3}{a^4}\frac{\left\{\sum_p \nu_p\left(\frac{M_p}{m_{\rm tot}}\right)^{2/3}\right\}^2 - 2\sum_{p\neq q}\nu_p\nu_q\left(\frac{M_p}{m_{\rm tot}}\right)^{2/3}\left(\frac{M_q}{m_{\rm tot}}\right)^{2/3}\sin^2(\theta_p-\theta_q)}{\sum_{p\neq q}\nu_p\nu_q - \sum_p \nu_p\left(\frac{M_p}{m_{\rm tot}}\right)^{2/3}}$$

$$f_{\rm GW}^2 = m_{\rm tot}/\pi^2 a^3$$

$$\frac{1}{f_{\rm GW}}\frac{df_{\rm GW}}{dt} = \frac{96}{5}\pi^{8/3}M_{\rm chirp}^{5/3}f_{\rm GW}^{8/3}$$

same as binary !



can be distinguished by higher moments

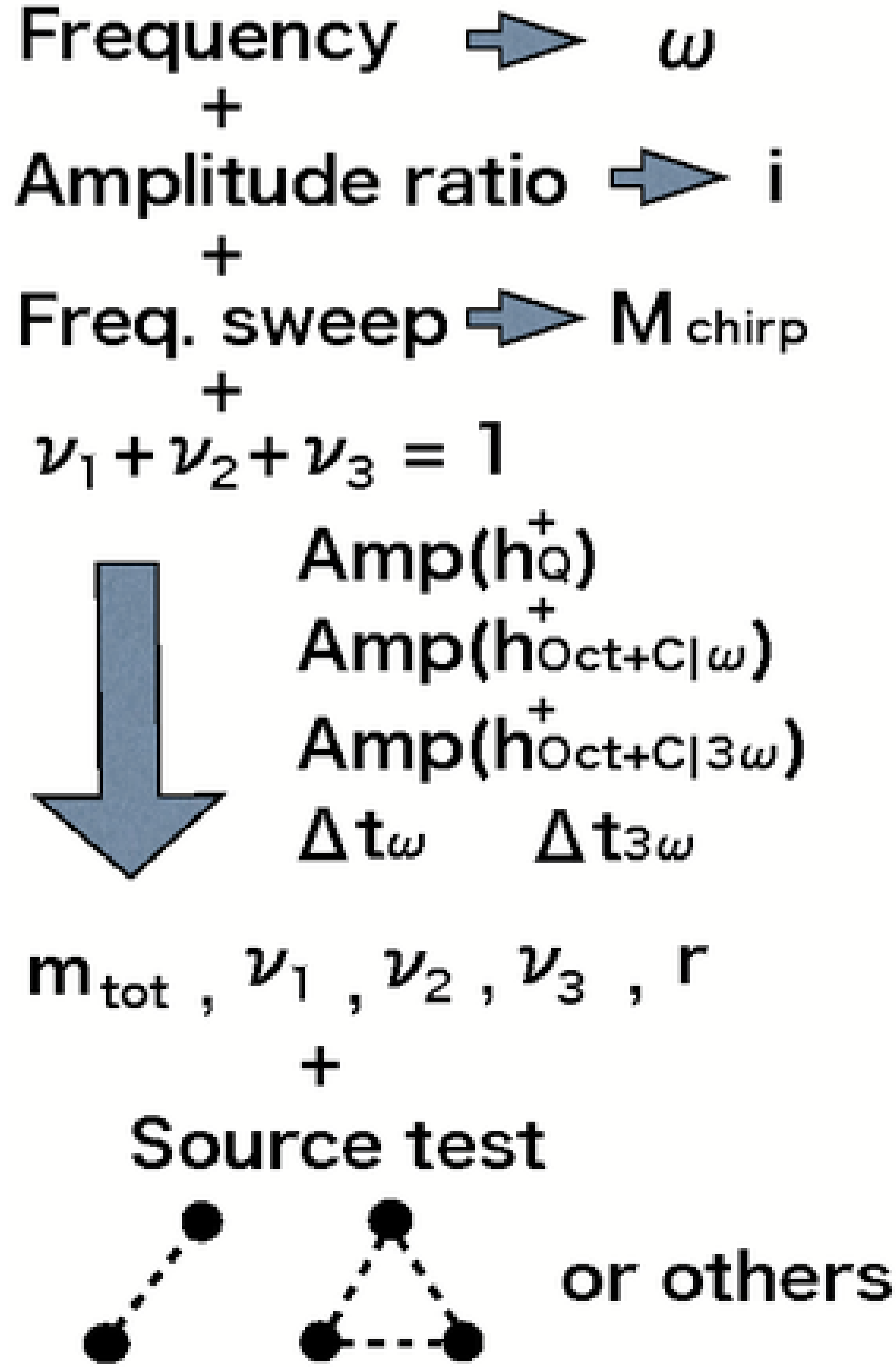
Torigoe et al. (2009), HA (2009)

Flow chart

Is GW source a binary?

Paramater determinations of particular 3-body

HA (2009)



§ 3 Summary

1. **Choreography in GR**

2. **GR extension of Euler+Lgrange**

Similarity and difference in Newtonian and GR sol.

A lot of interesting things to do!