1/R^2ポテンシャルの下での三体８の字解が満たす(ある)二階の斉次線形微分方程式について

三体は U(2) で舞う

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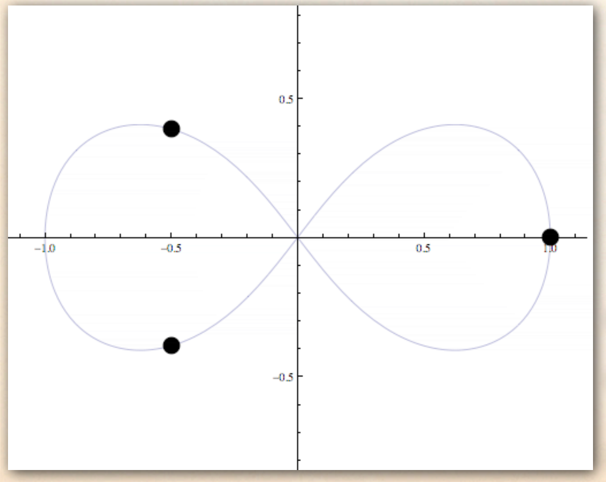
三体８の字解



$m_k = 1, k = 1, 2, 3,$

$\ddot{q}_k = \nabla_q V,$

$V = \frac{1}{\alpha} \sum \frac{1}{|q_i - q_j|^\alpha}$

$\sum q_k = 0,$

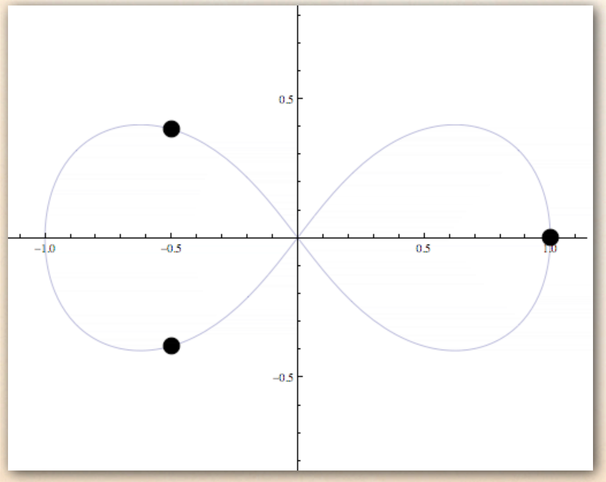
$\sum \dot{q}_k = 0.$

$q_1(t) = q(t),$

$q_2(t) = q\left(t + \frac{T_0}{3}\right),$

$q_3(t) = q\left(t + \frac{2T_0}{3}\right),$

1/R^2の下での三体８の字解



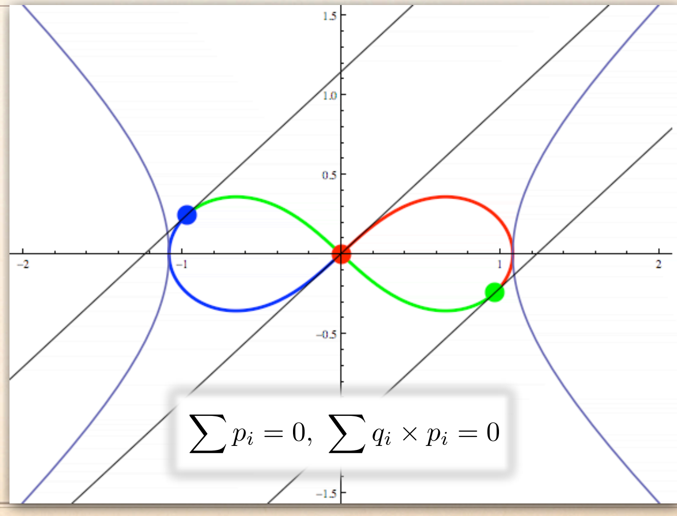
$C = \sum q_k \times p_k = 0.$

$V = \frac{1}{2} \sum \frac{1}{|q_i - q_j|^2}$

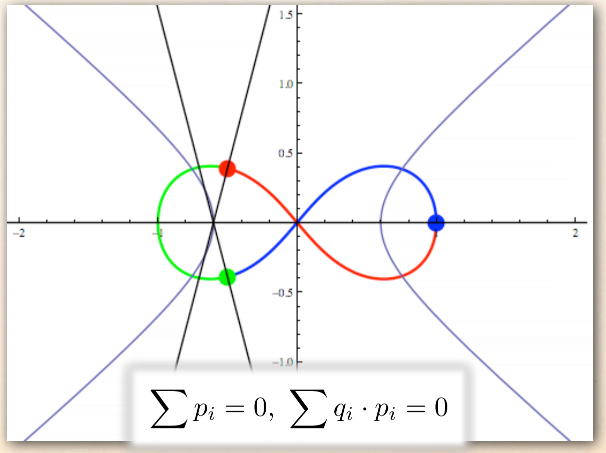
$\Downarrow$

$\dot{I} = \sum q_k \cdot p_k = 0.$

三接線定理

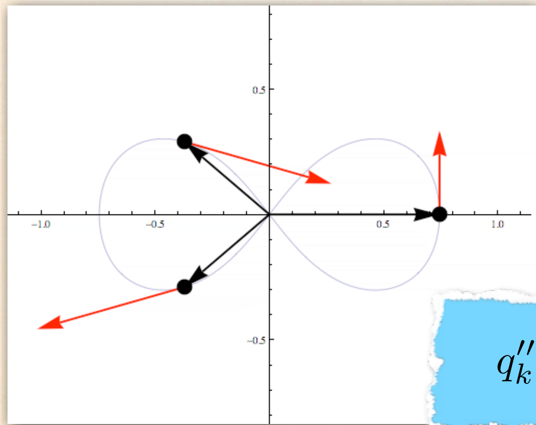


三法線定理



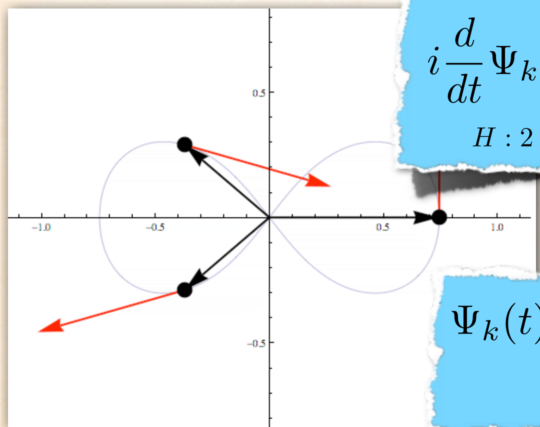
$\sum p_i = 0,\ \sum q_i \cdot p_i = 0$

（ある）二階線形微分方程式



$$q_k'' = i\phi' q_k' - \frac{1}{I} q_k$$

U(2)で舞う



$$i\frac{d}{dt}\Psi_k(t) = H(t)\Psi_k(t)$$

$H : 2 \times 2$ Hermite Matrix

↓

$$\Psi_k(t) = U(t,0)\Psi_k(0)$$

$$U(t,0) \in U(2)$$

$\dot{I} = 0,\ C = 0$ の意味

$$\dot{I} = 0, C = 0 \Rightarrow 0 = \dot{I} + iC = \sum (q_k \cdot p_k + i q_k \times p_k) = \sum q_k^* p_k.$$

$$\begin{aligned}\therefore 0 &= q_1^* p_1 + q_2^* p_2 + q_3^* p_3 \\ &= q_1^* p_1 + q_2^* p_2 + q_3^* (-p_1 - p_2) \\ &= -(q_3^* - q_1^*) p_1 + (q_2^* - q_3^*) p_2\end{aligned}$$

$$\therefore (q_3^* - q_1^*) p_1 = (q_2^* - q_3^*) p_2$$

$$\therefore \frac{p_1}{q_2^* - q_3^*} = \frac{p_2}{q_3^* - q_1^*} = \frac{p_3}{q_1^* - q_2^*} = \kappa e^{i\phi}$$

$\dot{I}=0,\ C=0$ の意味

$$\therefore \frac{p_1}{q_2^*-q_3^*}=\frac{p_2}{q_3^*-q_1^*}=\frac{p_3}{q_1^*-q_2^*}=\kappa e^{i\phi}$$

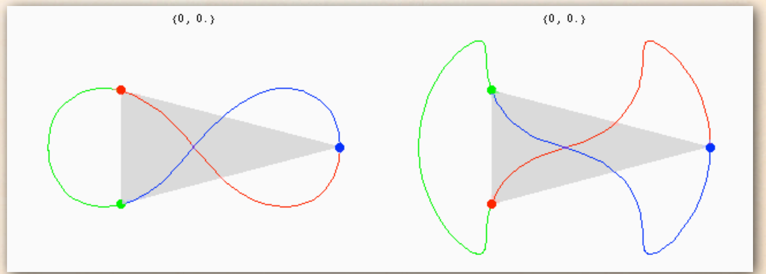
$$\therefore \sum |p_k|^2=\kappa^2\sum|q_i-q_j|^2=3\kappa^2\sum|q_k|^2$$

$$\therefore \kappa=\sqrt{\frac{K}{3I}} \qquad \left(K=\sum|p_k|^2,\ I=\sum|q_k|^2\right)$$

$$\therefore \frac{p_k}{\sqrt{K}}=\frac{1}{\sqrt{3I}}e^{i\phi}(q_i^*-q_j^*)$$

同期した相似三角形

$$\therefore \frac{p_k}{\sqrt{K}}=\frac{1}{\sqrt{3I}}e^{i\phi}(q_i^*-q_j^*) \quad\Leftrightarrow\quad \frac{q_k}{\sqrt{I}}=\frac{1}{\sqrt{3K}}e^{i\phi}(p_i^*-p_j^*)$$



$$\frac{q_k}{\sqrt{I}} \qquad\qquad \frac{p_i-p_j}{\sqrt{3K}}$$

同期した相似三角形

$$\frac{p_k}{\sqrt{K}}=\frac{1}{\sqrt{3I}}e^{i\phi}(q_i^*-q_j^*)$$

$ds=\sqrt{K}dt$ を使うと

$$\frac{p_k}{\sqrt{K}}=\frac{1}{\sqrt{K}}\frac{dq_k}{dt}=\frac{dq_k}{ds}=q_k'$$

$$\therefore q_k'=\frac{1}{\sqrt{3I}}e^{i\phi}(q_i^*-q_j^*)$$

二階微分方程式

$$\therefore q_k'=\frac{1}{\sqrt{3I}}e^{i\phi}(q_i^*-q_j^*)$$

もう一度微分して,

$$\therefore q_k''=i\phi' q_k'+\frac{1}{\sqrt{3I}}e^{i\phi}(q_i^{*\prime}-q_j^{*\prime})$$

$$\therefore q_k''=i\phi' q_k'-\frac{1}{I}q_k$$

（ある）二階微分方程式

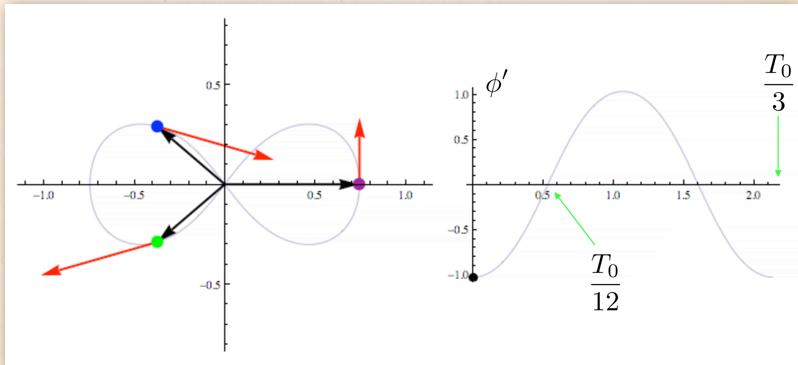
$$\therefore q_k'' = i\phi' q_k' - \frac{1}{I} q_k$$

ϕ' をgivenとみなせば，線形微分方程式

$q_k, (k = 1, 2, 3)$ は，同じ微分方程式の解

（ある）二階微分方程式

$$\therefore q_k'' = i\phi' q_k' - \frac{1}{I} q_k$$



二階微分方程式

$$\therefore q_k'' = i\phi' q_k' - \frac{1}{I} q_k$$

Equations of motion $\ddot{q}_k = \frac{\partial}{\partial q_k} V$ ············▶ q_k

? ? ?

Equation of motion for ϕ' ? → ϕ'

二成分にして一階に

$$\therefore q_k'' = i\phi' q_k' - \frac{1}{I} q_k$$

$$\Psi_k = \begin{pmatrix} \frac{q_k}{\sqrt{I}} \\ q_k' \end{pmatrix}$$ とおくと，

$$\frac{d\Psi_k(s)}{ds} = A(s)\Psi_k(s), \text{ with } A(s) = \begin{pmatrix} 0 & \frac{1}{\sqrt{I}} \\ -\frac{1}{\sqrt{I}} & i\phi'(s) \end{pmatrix}$$

$A^\dagger = -A$ 反エルミート

二成分にして一階に

あるいは，エルミート行列

$$H_A = iA = \begin{pmatrix} 0 & \frac{i}{\sqrt{I}} \\ -\frac{i}{\sqrt{I}} & -\phi'(s) \end{pmatrix} = -\frac{\sigma_2}{\sqrt{I}} - \frac{1-\sigma_3}{2}\phi'$$

σ_k : Pauli spin matrices

を使って，

$$i\frac{d}{ds}\Psi_k = H_A\Psi_k$$

$$ds = \sqrt{K}dt \Rightarrow i\frac{d}{dt}\Psi_k = \sqrt{K}\ H_A\Psi_k$$

ユニタリー行列

そこで，

$$i\frac{d}{ds}U(s,0) = H_A(s)U(s,0), \text{ with } U(0,0) = 1$$

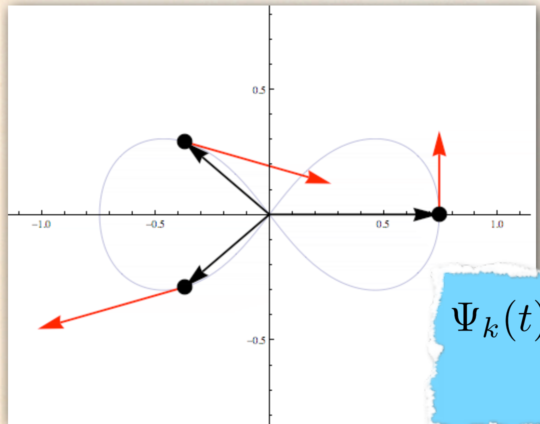
で $U(s,0)$ を定義すると，

$$\Psi_k(s) = U(s,0)\Psi_k(0)$$

H_A :Hermite $2\times 2 \ \Rightarrow\ U(s,0) \in U(2)$

$$\det(U(s,0)) = \exp\left(-i\int_0^s ds_1\ \mathrm{tr}(H_A(s_1))\right) = \exp\left(i\phi(s) - i\phi(0)\right)$$

三体はU(2)で舞う



$q_k \in \mathbb{C}$,

$$\Psi_k(t) = \begin{pmatrix} \frac{q_k}{\sqrt{\sum |q_k|^2}} \\ \frac{\dot{q}_k}{\sqrt{\sum |\dot{q}_k|^2}} \end{pmatrix}$$

$$\downarrow$$

$$\Psi_k(t) = U(t,0)\Psi_k(0)$$

$$U(t,0) \in U(2)$$

保存量

$$\Psi_i^\dagger(s)\Psi_j(s) = \Psi_i^\dagger(0)U^\dagger(s,0)U(s,0)\Psi_j(0) = \Psi_i^\dagger(0)\Psi_j(0)$$

$$\Psi_k = \begin{pmatrix} \frac{q_k}{\sqrt{I}} \\ q_k' \end{pmatrix} = \begin{pmatrix} \frac{q_k}{\sqrt{I}} \\ \frac{\dot{q}_k}{\sqrt{K}} \end{pmatrix}$$ なので，これは

$$\frac{q_i^* q_j}{I} + \frac{\dot{q}_i^*\dot{q}_j}{K} = \begin{cases} \frac{2}{3} & \text{for } i = j, \\ -\frac{1}{3} & \text{for } i \neq j. \end{cases}$$

保存量

$$\frac{q_i^* q_j}{I}+\frac{\dot q_i^* \dot q_j}{K}=\begin{cases}\frac{2}{3} & \text{for } i=j,\\ -\frac{1}{3} & \text{for } i\neq j.\end{cases}$$

$$\frac{|q_k|^2}{I}+\frac{|\dot q_k|^2}{K}=\frac{2}{3}$$

$$\frac{q_i\cdot q_j}{I}+\frac{\dot q_i\cdot \dot q_j}{K}=-\frac{1}{3}\quad \text{for } i\neq j$$ New!

$$\frac{q_i\times q_j}{I}+\frac{\dot q_i\times \dot q_j}{K}=0$$

拡張

For planar $q_k(t)$, with $\dot I\neq 0, C\neq 0$ and $\forall m_k$

$$Q_k(t)=\exp\left(-i\int_0^t \frac{C}{I}dt\right)\frac{q_k}{\sqrt{I}}$$

$$I=\sum m_k|Q_k|^2=1,\ K=\sum m_k|\dot Q_k|^2$$

$$\sum Q_k\cdot m_k\dot Q_k=0, \sum Q_k\times m_k\dot Q_k=0,$$

$$Q_k''=i\phi' Q_k'-Q_k$$

ここまでは，．．．

$\dot I=0,\ C=0$　の運動一般の話

三体８の字解の場合は，周期的　$q(t)=q\left(t+T_0\right)$

$$q_1(t)=q(t),$$

$$q_2(t)=q\left(t+\frac{T_0}{3}\right),$$

$$q_3(t)=q\left(t+\frac{2T_0}{3}\right)$$

$\Rightarrow$ $U(t,0)$　の周期性は？　　$q_i(t)$ と $q_j(t)$ の関係は？

三体８の字解の場合

$\phi'\left(s+s_0\right)=\phi'\left(s\right)$　　s_0 :1/3周期

$$H_A(s)=\begin{pmatrix}0 & \frac{i}{\sqrt{I}}\\ -\frac{i}{\sqrt{I}} & -\phi'(s)\end{pmatrix}\Rightarrow H_A(s+s_0)=H_A(s)$$

$i\frac{d}{ds}U(s,0)=H_A(s)U(s,0)$　$\Rightarrow$　$U(s+s_0,s)=\Omega(s)$ と置くと，

$$\det(U(s+s_0,s))=\exp\left(i\phi(s+s_0)-i\phi(s)\right)=1$$

$\Omega(s)\in SU(2)$, $i\frac{d}{ds}\Omega(s)=H_A(s)\Omega(s)-\Omega(s)H_A(s)$

三体の関係はSPECIAL

$U(s+s_0,s)=\Omega(s)$

$$\Psi_2(s)=\Psi_1(s+s_0)=U(s+s_0,s)\Psi_1(s)=\Omega(s)\Psi_1(s),$$
$$\Psi_3(s)=\Psi_2(s+s_0)=\Omega(s)\Psi_2(s)=\Omega(s)^2\Psi_1(s)$$

$$\therefore 0=\Psi_1(s)+\Psi_2(s)+\Psi_3(s)=\left(1+\Omega(s)+\Omega(s)^2\right)\Psi(s)$$

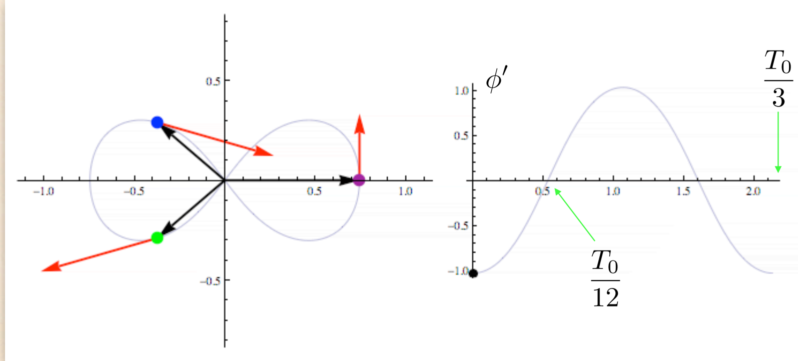
従って，$\Omega(s)$ の固有値は，1の３乗根 ω,ω^2

しかし，これらは今のところ

何も新しい条件を与えない．．．

まとめ

$$q_k''=i\phi' q_k'-q_k,\ \ \Psi_k(t)=U(t,0)\Psi(0),\ \ \phi'\ ?$$



ありがとうございました