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# Choreograpies in the Symmetric Four-Vortex Problem

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**Abstract.** We qualitatively studied the four-vortex system which maintains symmetry with respect to the origin. We completely classified the solutions in the two simple cases. There exist double choreographies.

## 1. Introduction

Motion of vortices in $\mathbb{R}^2$ is described by the following differential equations

$$\dot{z}_k = i \sum_{j \neq k} \frac{\Gamma_j}{\overline{z}_k - \overline{z}_j}, \qquad (k = 1, 2, \cdots, N) \tag{1}$$

where $i = \sqrt{-1}$, $z_k = x_k + iy_k \in \mathbb{C}$ denotes the position of the $k$-th vortex, $\dot{z}_k = dz_k/dt$, and $\Gamma_k \in \mathbb{R}\backslash\{0\}$ the $k$-th vorticity divided by $2\pi$. In general, the cases for $N \leq 3$ are integrable, and they have been studied well [1–5]. In this paper, we study a special case where four vorticies are kept symmetry with respect to the origin. Namely, the system satisfies the following conditions through motion.

$$\Gamma_3 = \Gamma_1,\ \Gamma_4 = \Gamma_2,\ z_3 = -z_1,\ z_4 = -z_2 \tag{2}$$

The equations of motion (1) is reduced to

$$\begin{cases} \dot{z}_1 = i \left( \dfrac{\Gamma_1}{2\overline{z}_1} + \dfrac{\Gamma_2}{\overline{z}_1 - \overline{z}_2} + \dfrac{\Gamma_2}{\overline{z}_1 + \overline{z}_2} \right), \\ \dot{z}_2 = i \left( \dfrac{\Gamma_2}{2\overline{z}_2} + \dfrac{\Gamma_1}{\overline{z}_2 - \overline{z}_1} + \dfrac{\Gamma_1}{\overline{z}_2 + \overline{z}_1} \right). \end{cases} \tag{3}$$

The dynamical system (3) permits two integrals:

$$\begin{cases} I = \Gamma_1(x_1^2 + y_1^2) + \Gamma_2(x_2^2 + y_2^2), \\ h = |z_1|^{\Gamma_1^2} |z_2|^{\Gamma_2^2} (|z_1 - z_2|\ |z_1 + z_2|)^{2\Gamma_1\Gamma_2}, \end{cases} \tag{4}$$

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where $I$ and $\tilde{h} = \log h$ should be called the *moment of inertia* and the *energy*, respectively. Therefore, the dynamical system (3) is integrable. In principle, the solutions can be obtained by quadrature. Almost all solutions are periodic or quasi-perodic. The behaviour is not much complicated. It, however, has not yet enough studied until now.

We find the following theorems in the cases of $\Gamma_1 > \Gamma_2 > 0$ and $\Gamma_1 > 0 > \Gamma_2$.

**Theorem 1** There are choreographic solutions, quasi-periodic solutions, homographic solutions, heteroclinic connections, but no collisions.

**Theorem 2** When $\Gamma_1 \geq \Gamma_2 > 0$, there are two different regular tetragon homographic solutions which are stable, and two different collinear homographic solutions which are unstable.

**Theorem 3** When $\Gamma_1 > 0 > \Gamma_2$, there are two different rhomboidal homographic solutions which are unstable, but no collinear homographic solutions. There are heteroclinic connections between the rhomboidal homographic solutions to infinity.

Theorem 2 and 3 will be shown in the following section, individually. Theorem 1 will be shown as the summary of them.

## 2. Symmetric Four-Vortex Problem

### 2.1. Equations of motion in the polar form

Here we introduce the polar form: $z_k = r_k e^{i\theta_k}$. The equations of motion are transformed to as follows.

$$\dot{r}_1 = \frac{2\Gamma_2 r_1 r_2^2 \sin 2\theta}{(r_1^2 + r_2^2)^2 - 4r_1^2 r_2^2 \cos^2\theta}, \quad \dot{r}_2 = -\frac{2\Gamma_1 r_1^2 r_2 \sin 2\theta}{(r_1^2 + r_2^2)^2 - 4r_1^2 r_2^2 \cos^2\theta}. \quad (5)$$

If we introduce new set of variables:

$$\theta = \theta_2 - \theta_1, \quad \psi = \frac{\theta_1 + \theta_2}{2},$$

then we have

$$\begin{cases} \dot{\theta} = \dfrac{\Gamma_2 r_1^2 - \Gamma_1 r_2^2}{2r_1^2 r_2^2} + \dfrac{2\left(\Gamma_1 r_2^2 - \Gamma_2 r_1^2 - (\Gamma_1 r_1^2 - \Gamma_2 r_2^2)\cos 2\theta\right)}{(r_1^2 + r_2^2)^2 - 4r_1^2 r_2^2 \cos^2\theta}, \\ \dot{\psi} = \dfrac{\Gamma_2 r_1^2 + \Gamma_1 r_2^2}{4r_1^2 r_2^2} + \dfrac{\Gamma_1 r_2^2 + \Gamma_2 r_1^2 - I\cos 2\theta}{(r_1^2 + r_2^2)^2 - 4r_1^2 r_2^2 \cos^2\theta}, \end{cases} \quad (6)$$

where $\theta \in (-\pi, \pi]$ and $\psi \in (-\pi, \pi]$, and the integrals in the new variables

$$\begin{cases} I = \Gamma_1 r_1^2 + \Gamma_2 r_2^2, \\ h = r_1^{\Gamma_1^2} r_2^{\Gamma_2^2} \left\{(r_1^2 + r_2^2)^2 - 4r_1^2 r_2^2 \cos^2\theta\right\}^{\Gamma_1\Gamma_2}. \end{cases} \quad (7)$$

The right hand sides of Eq. (4) and (5) do not contain $\psi$. This means that the variable $\psi$ is separable. By using the integrals $I$ and $h$,

### 2.2. Case 1: $\Gamma_1 \geq \Gamma_2 > 0$

Let $\Gamma_1 = 1/\alpha^2$ and $\Gamma_2 = 1/\beta^2$ with $\beta \geq \alpha > 0$. Transform variables

$$r_1 = \alpha \cos\varphi, \quad r_2 = \beta \sin\varphi,$$

and we have $I = 1$. Then, the energy becomes

$$h = \alpha^{\alpha^{-4}} \beta^{\beta^{-4}} \cos^{\alpha^{-4}}\varphi \sin^{\beta^{-4}}\varphi((\alpha^2\cos^2\varphi + \beta^2\sin^2\varphi)^2 - \alpha^2\beta^2\sin^2 2\varphi\cos^2\theta)^{\alpha^{-2}\beta^{-2}}.$$

Instead of $h$, we will study $\alpha^{-\alpha^{-4}}\beta^{-\beta^{-4}}h$, namely

$$f(\theta,\varphi) = \cos^{\alpha^{-4}}\varphi \sin^{\beta^{-4}}\varphi((\alpha^2\cos^2\varphi + \beta^2\sin^2\varphi)^2 - \alpha^2\beta^2\sin^2 2\varphi\cos^2\theta)^{\alpha^{-2}\beta^{-2}}.$$

### 2.3. Case 2: $\Gamma_1 > 0 > \Gamma_2$

Let $\Gamma_1 = 1/\alpha^2$ and $\Gamma_2 = -1/\beta^2$. Transform variables

$$r_1 = \alpha \cosh\varphi, \quad r_2 = \beta \sinh\varphi,$$

and we have $I = 1$. Then, the energy becomes

$$h = \frac{\alpha^{\alpha^{-4}}\beta^{\beta^{-4}}\cosh^{\alpha^{-4}}\varphi \sinh^{\beta^{-4}}\varphi}{((\alpha^2\cosh^2\varphi + \beta^2\sinh^2\varphi)^2 - \alpha^2\beta^2\sinh^2 2\varphi\cos^2\theta)^{2\alpha^{-2}\beta^{-2}}}.$$

Instead of $h$, we will study $\alpha^{-\alpha^{-4}}\beta^{-\beta^{-4}}h$, namely

$$g(\theta,\varphi) = \frac{\cosh^{\alpha^{-4}}\varphi \sinh^{\beta^{-4}}\varphi}{((\alpha^2\cosh^2\varphi + \beta^2\sinh^2\varphi)^2 - \alpha^2\beta^2\sinh^2 2\varphi\cos^2\theta)^{2\alpha^{-2}\beta^{-2}}}.$$

### 2.4. Example 1: $\Gamma_1 = \Gamma_2 = \gamma > 0$

In this case, the integrals become

$$I = \gamma(r_1^2 + r_2^2), \quad h = [r_1 r_2 \left\{(r_1^2 + r_2^2)^2 - 4r_1^2 r_2^2\cos^2\theta\right\}]^{\gamma^2}.$$

Assuming $I = \gamma$, let

$$r_1 = \cos\varphi, \quad r_2 = \sin\varphi.$$

It is sufficient to take $0 < \varphi < \frac{\pi}{2}$ because $r_1 > 0$, $r_2 > 0$. If we permit binary collision of vorteces, then we extend it to $0 \leq \varphi \leq \frac{\pi}{2}$. Relations $(\theta,\varphi) = \left(0, \frac{\pi}{4}\right)$, $\left(\pi, \frac{\pi}{4}\right)$ also express binary collision. By using these variables, energy can be expressed as

$$h = \frac{1}{2}\sin 2\varphi(1 - \sin^2 2\varphi\cos^2\theta). \tag{8}$$

Then, the equations of motion become

$$\dot{\theta} = 2\gamma \cot 2\varphi \left( \frac{1}{\sin 2\varphi} - \frac{2 \sin 2\varphi \cos^2\theta}{1-\sin^2 2\varphi \cos^2\theta} \right), \tag{9}$$

$$\dot{\varphi} = -\frac{\gamma \sin 2\varphi \sin 2\theta}{1-\sin^2 2\varphi\cos^2\theta}, \tag{10}$$

$$\dot{\psi} = \frac{\gamma}{\sin 2\varphi}\left( \frac{1}{\sin 2\varphi} + \frac{\sin 2\varphi - \cos\theta}{1-\sin 2\varphi\cos\theta} + \frac{\sin 2\varphi + \cos\theta}{1+\sin 2\varphi\cos\theta} \right). \tag{11}$$

The phase space is described in terms of three variables $(\theta, \psi, \varphi)$, and homeomorphic to $S^1 \times S^1 \times [0, \pi/2]$. Since $h$ does not include $\psi$, contours of $h$ are projection of solution curves $(\theta(t), \psi(t), \varphi(t))$. Now let us consider $h(\theta, \varphi)$ as a function of $\theta$ and $\varphi$. The extrema of $h(\theta, \varphi)$ are summerized in Table 1 by simple calculations. The contours of $h(\theta, \varphi)$ are illustrated in Figure 1. This contour map clarifies that the solutions are in general

Table 1. Extrema of $h(\theta, \varphi)$

$\alpha = \frac{1}{2}\arcsin\frac{1}{\sqrt{3}} \in (0, \frac{\pi}{2})$, and $\beta = \frac{\pi}{2} - \alpha$.

| $(\theta, \varphi)$ | $h(\theta, \varphi)$ | description |
|---|---|---|
| $\left(0, \frac{\pi}{4}\right)$, $\left(\pi, \frac{\pi}{4}\right)$ | $0$ | local minimum<br>binary collision |
| $\left(\pm\frac{\pi}{2}, \frac{\pi}{4}\right)$ | $\frac{1}{2}$ | local maximum<br>square homographic solution |
| $(0, \alpha)$, $(0, \beta)$, $(\pi, \alpha)$, $(\pi, \beta)$ | $\frac{\sqrt{3}}{9}$ | saddle<br>collinear homographic solution |

periodic in $(\theta, \varphi)$ except for the cases indicated in Table 1. The solutions are also periodic in $\psi$ except for binary collision. Let $T_1$ be a period in $(\theta, \varphi)$, namely

$$T_1 = 2\int_{\varphi_{min}}^{\varphi_{max}} \frac{d\varphi}{\dot{\varphi}} = \int_{\varphi_{min}}^{\varphi_{max}} \frac{\sin^2 2\varphi \cos^2\theta - 1}{\gamma \sin 2\varphi \sin 2\theta}\, d\varphi$$

and let $T_2$ be a half period in $\psi$, namely

$$T_2 = \int_0^{\pi/2} \frac{d\psi}{\dot{\psi}} = \int_0^{\pi/2} \frac{(1-\sin 2\varphi\cos\theta)(1+\sin 2\varphi\cos\theta)\sin^2 2\varphi}{\gamma(1+\sin 2\varphi)(1+\sin^2 2\varphi - (1+\cos^2\theta)\sin 2\varphi)}\, d\psi.$$

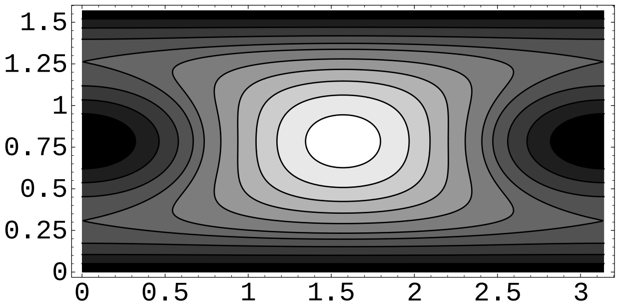


Figure 1. Contours of $h(\theta, \varphi)$ with abscissa $\theta \in [0, \pi]$ and ordinate $\varphi \in \left[0, \frac{\pi}{2}\right]$

We can assume $\frac{\alpha}{\beta}$ irreducible for $\alpha, \beta \in \mathbb{N}$. If $\alpha T_1 = \beta T_2$, then the solution is periodic. Moreover if $\beta$ is odd number, the solution is choreography. If $T_1/T_2$ is irrational, then it is quasi-periodic.

**Theorem 1 :** In the equi-vortex symmetric four-votex system (5,6,7),

- Square homographic solution is stable.
- Collinear homographic solution is unstable.
- There exist heteroclinic solutions connecting two different collinear homographic solutions.
- All solutions other than heteroclinic solutions are periodic or quasi-periodic. Among them double choreography exists.
- No solutions begin from (end at) binary collision (Binary collision is irregularizable).

### 2.5. Example 2: $\Gamma_1 = -\Gamma_2 = \gamma > 0$

In this case, the integrals become

$$\begin{aligned} I &= \gamma(r_1^2 - r_2^2), \\ h &= \frac{r_1 r_2}{(r_1^2 + r_2^2)^2 - 4r_1^2 r_2^2 \cos^2\theta}. \end{aligned}$$

Moreover introduce the following transformation by letting $I = \gamma$.

$$r_1 = \cosh\varphi, \quad r_2 = \sinh\varphi.$$

It is sufficient to take $\varphi > 0$ because $r_1 > 0$, $r_2 > 0$.

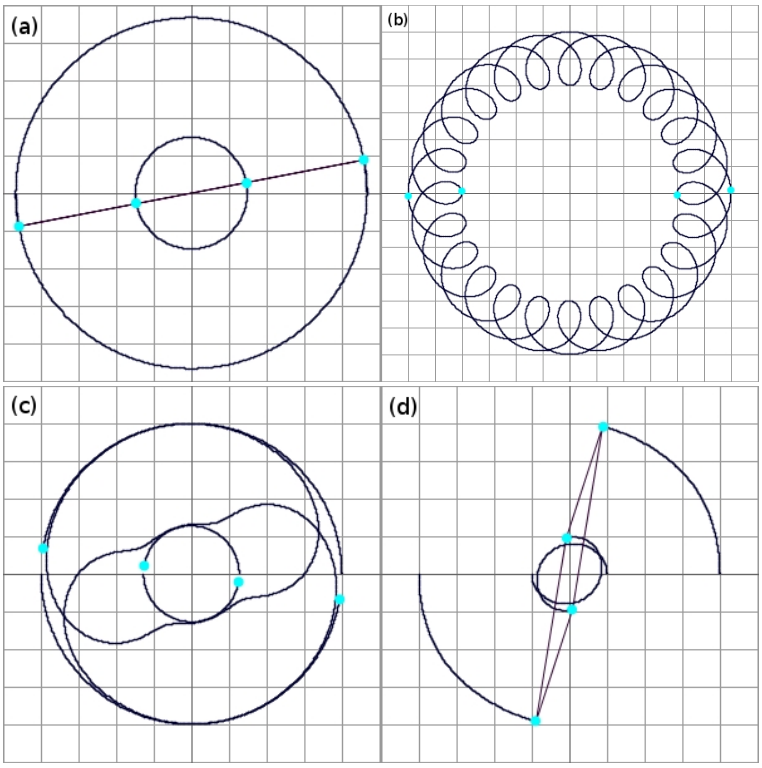


Figure 2. (a) collinear homographic solution, (b) double choreography, (c,d) two different solutions in the neighborhood of heteroclinic solutions

Equations of motion with $h$ are rewritten to the following expressions by using new variables.

$$\dot{\theta} = \frac{2\gamma(\sinh^2 2\varphi \sin^2\theta - 1)\cosh 2\varphi}{(1+\sinh^2 2\varphi \sin^2\theta)\sinh^2 2\varphi} \tag{12}$$

$$\dot{\varphi} = -\frac{\gamma \sinh 2\varphi \sin 2\theta}{1+\sinh^2 2\varphi \sin^2\theta} \tag{13}$$

$$\dot{\psi} = -\frac{\gamma(\cosh^2 2\varphi + \sinh^2 2\varphi \cos^2\theta)}{(1+\sinh^2 2\varphi \sin^2\theta)\sinh^2 2\varphi} \tag{14}$$

$$h = \frac{\sinh 2\varphi}{2(1+\sinh^2 2\varphi \sin^2\theta)}. \tag{15}$$

The phase space is described in terms of three variables $(\theta, \psi, \varphi)$, and homeomorphic to $S^1 \times S^1 \times \mathbb{R}^+$. Since $h$ does not include $\psi$, contours of $h$ are projection of solution curves $(\theta(t), \psi(t), \varphi(t))$. Now let us consider $h(\theta, \varphi)$ as a function of $\theta$ and $\varphi$. The extrema of $h(\theta, \varphi)$ are summerized in Table 2 by simple calculations. The contours of $h(\theta, \varphi)$ are illustrated in Figure 3.

Table 2. Extrema of $h(\theta, \varphi)$ for $\Gamma_1 = -\Gamma_2 = \gamma > 0$

| $(\theta, \varphi)$ | $h(\theta, \varphi)$ | description |
|---|---|---|
| $(*, 0)$ | $0$ | local minimum<br>binary collision |
| $\left(\pm\frac{\pi}{2}, \frac{1}{2}\ln(1+\sqrt{2})\right)$ | $\frac{1}{4}$ | saddle<br>rhomboidal homographic solution |

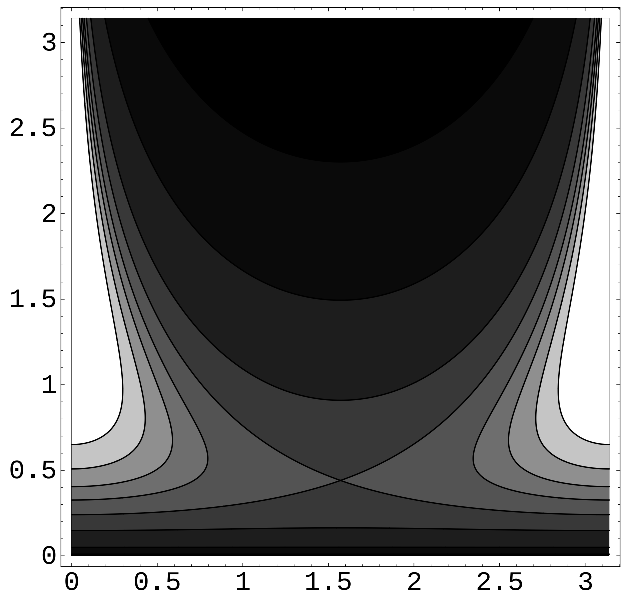


Figure 3. Contours of $h(\theta, \varphi)$ for $\Gamma_1 = -\Gamma_2 = \gamma > 0$ with absissa $\theta \in [0, \pi]$ and ordinate $\varphi \in [0, \pi]$

Elementary calculations clarify no local maxima in $h(\theta, \varphi)$. In Figure 3, solutions below the bounded heteroclinic solution are periodic in $\theta$ while one above it excape to infinity as $t \to \pm\infty$. On the other hand, any solution is periodic in $\psi$. We can assume $\dfrac{\alpha}{\beta}$ irreducible for $\alpha, \beta \in \mathbb{N}$. If $\alpha T_1 = \beta T_2$, then the solution is periodic. Moreover if $\beta$ is odd number, the solution is choreography. If $T_1/T_2$ is irrational, then it is quasi-periodic.

**Theorem 2 :** In the anti-equi-vortex symmetric four-votex system (8,9,10),

- There never exist square homographic solution and collinear homographic solution
- Rhomboidal homographic solution is unstable.
- There exist bounded heteroclinic solutions connecting two different rhomboidal homographic solutions.
- There exist unbounded heteroclinic solutions connecting rhomboidal homographic solutions and infinity.
- All solutions other than heteroclinic solutions and escapers are periodic or quasi-periodic. Among them double choreography exists.
- No solutions begin from (end at) binary collision
  (Binary collision is irregularizable).

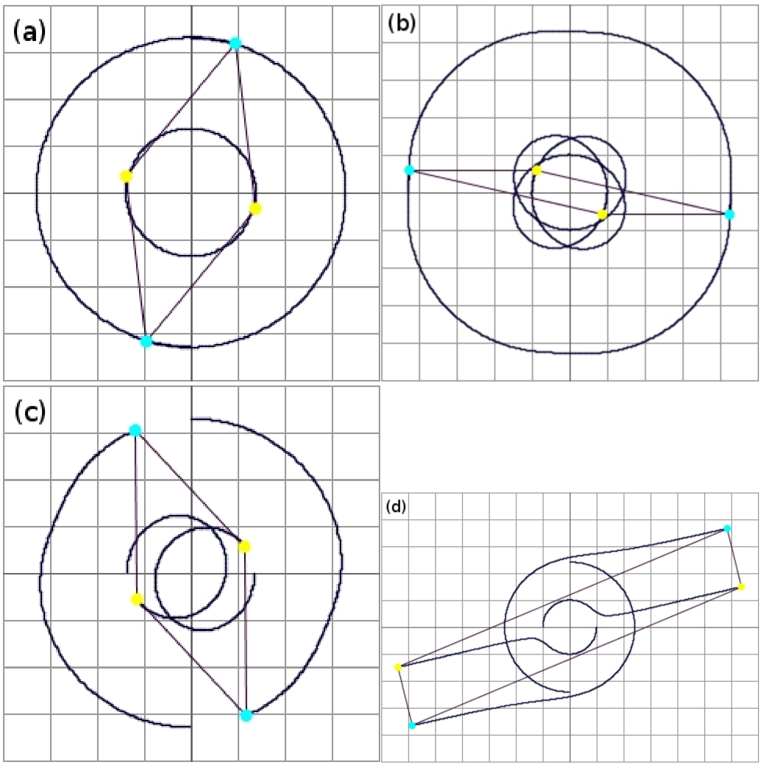


Figure 4. (a) rhomboidal homographic solution, (b) double choreography, (c,d) two different solutions in the neighborhood of heteroclinic solutions

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