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A dynamical system approximation to the analytical modeling of the roto–translatory coupling of the Earth internal motions

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Abstract. We construct an analytical model to describe the internal translational motion of a three layer body. This body is assumed to be composed of three homogeneous constituents: a rigid spherical shell enclosing a perfect fluid with a rigid spherical body. By means of a dynamical system approach based on Lagrangian mechanics, we compute the equations of motion of the system and its solution in the small oscillations approximation. This method allows us to obtain a clear analytical representation of the dynamics. As an application of the theory, we calculate the characteristics of the internal translational motion of some three layer models of the Earth, Mercury, and the icy bodies Europa, Titania, Oberon, Triton, and Pluto.

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1. Introduction

The classical celestial mechanics approach to model the motion of extended bodies separates it in two parts: one aims at determining the motion of the barycenter of the body. The other one is concerned with the relative motion of the body about its barycenter.

In general, the relative motion must be described by giving the temporal and spatial dependence of a vectorial field, like the displacement vector of an elastic material or the velocity field of a fluid. The rigid motion components associated to these fields, i.e. the relative translations or rotations of some constituents of the body, are specially interesting due, among others, to their influence in the definition of the reference systems linked to the body. These motions depends drastically on the internal structure

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of the body. For example, the rotational motion of a rigid body is quite different from the rotation of an ellipsoidal shell filled with a fluid.

From this perspective, one mathematical model particularly interesting is a three layer body composed of an external solid shell containing a fluid with a solid (Figure 1). Even in the simplified case when the solids are rigid bodies and the fluid is perfect, the internal dynamics of this system is very rich since the solid constituents can perform both relative rotations and translations. The investigation of these rotational and translational internal motions, and their couplings, as well as their interactions with the motion of the barycenter of the whole body, represents a challenging problem.

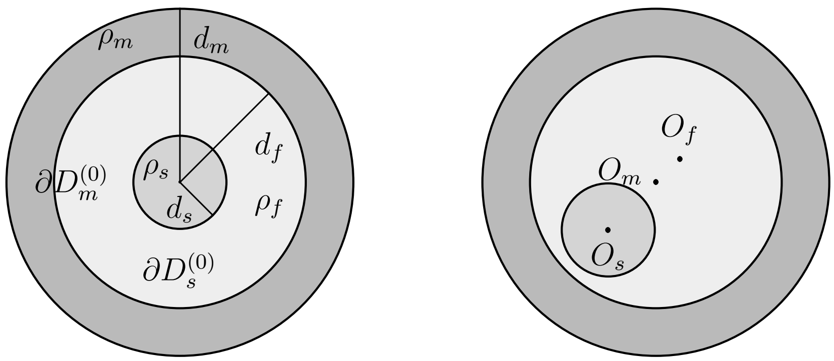

Figure 1. Reference (left panel) and instantaneous configuration (right panel) of the three layer model considered in this work

In addition to its dynamical interest, the relevance of this investigation is amplified if we take into account that this three layer model reproduces to some extent the real structure of some celestial bodies. It is the case of the Earth, although other bodies (e.g. Grinfeld & Wisdom 2005, Hussmann et al. 2006) like Mercury or the medium–sized icy bodies Europa, Titan Oberon, Triton, and Pluto might present this structure. The observation of their internal rigid motions, or of the effects induced by them, could provide some indirect mean to constrain their inner structure.

The complexity of the problem forces to develop a successive approximations scheme in order to construct a reliable mathematical model of this dynamical system. In this sense, a deep understanding of the roto–translatory coupling of the rigid internal motions requires first to consider separately the rotational and translational problems. In this note we will be concerned with the internal translational motion in the dynamical system framework, providing an sketch of the analytical treatment of this problem. A comprehensive development of it can be found in Escapa & Fukushima (2010).

At this stage, we are going to consider a simplified model of a three layer body that describes the main features of the dynamics. Let us recall that, with the exception of the Earth, the internal structure of other celestial bodies is not well determined, so it would not have much sense to develop a more complex model involving unknown rheological parameters. In particular, the model is assumed to be composed of three layers of constant density (Figure 1): the mantle with density ρ_m and mass m_m, the fluid outer core with density ρ_f and mass m_f, and the solid inner core with density ρ_s and mass m_s. The mantle is a rigid spherical shell with internal radius d_f and external d_m, and the solid inner core a rigid sphere of radius d_s. The fluid is perfect, i.e. without viscosity and incompressible. The evolution of the system is a consequence of the gravitational interactions among the layers and the hydrodynamical interaction exerted by the fluid on the solid constituents.

A close model to that described previously was worked out by Slichter (1961). His study was motivated by the possibility of detecting the internal translational motion of the Earth inner core after a great earthquake episode. With heuristical arguments, Slichter showed that the motion of the system is similar to that of a harmonic oscillator with frequency ω, usually referred as Slichter mode. This frequency depends on the rheological parameters of the Earth interior: density of the fluid, of the inner core, etc. Later, some analytical investigations of this problem were also performed (see Escapa & Fukushima 2010) like the works of Busse (1974) or Grinfeld & Wisdom (2005).

Here, we will consider the same problem as that posed in Grinfeld & Wisdom (2005) but approaching it with different techniques and providing also some different application. In this problem it is assumed that solid layers move only translationally, the fluid motion being generated by the motion of the rigid constituents. The objective is to determine the nature of the dynamics, especially to obtain an analytical expression of the frequency ω in terms of the physical properties of the three layer body. This expression will allow us to understand the influence of the motion of the mantle in ω, since in the works of Slichter (1961) and Busse (1974) this layer was assumed to be in rest. This assumption was taken as a consequence of the small size of the Earth inner core with respect to the mantle. However, as it was pointed out in Grinfeld & Wisdom (2005), it is not the case of Mercury that has a large inner core. Hence, the interest of determining the contributions of the motion of the mantle to ω. The analysis performed in Grinfeld & Wisdom (2005) showed that, as in the Earth case, also for Mercury that influence is negligible, although these authors could not find a clear explanation of this fact. Our approach will allow us to explain well the origin of this circumstance. In addition, we will also show that the value of ω depends significatively on the different possible internal structures of some icy bodies. Hence, the observation of this frequency might provide another complementary way to constrain the interior of these bodies, which is poorly known.

2. Dynamical approach

In contrast to other approaches (e.g. Busse 1974, Grinfeld & Wisdom 2005) the dynamics of the system will be determined through a variational principle. Within this framework the equations of the motion are derived from the kinetic energy, the potential energy, and the generalized forces of the system. The main advantages of this method are that the solid and fluid layers are treated as one single dynamical system, and that it is possible to apply the celestial mechanics tools to study internal translational and rotational motions dependencies, as well as the possible coupling mechanisms with the external motions.

Another important point is that the hydrodynamical interaction among the fluid and the solids is automatically incorporated in the kinetic energy of the system (Lamb 1963, chp. VI). Hence, it is not necessary the calculation of the effect of the fluid pressures on the surfaces of the solids. This approach has also been employed successfully in the study of the rotational motion of a three layer Earth model (e.g. Escapa et al. 2003).

In the case of our model we will construct the equations of motion from the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_i}\right) = 0, \tag{1}$$

where the Lagrangian function is the difference between the kinetic and potential energies

$$\mathcal{L} = \mathcal{T} - \mathcal{V}. \tag{2}$$

In this method the dynamical configuration of the system is specified through the generalized coordinates and velocities q_i and $\dot{q}_i$, respectively. For our model it can be shown (Escapa & Fukushima 2010) that this dynamical system has only three degrees of freedom, that is to say, we only need three independent coordinates q_i, and their associated velocities, to describe its time evolution.

This fact is a consequence of considering only the translational motion, and the particular symmetry of our system that allows us to link the evolution of the barycenter of the three constituents of the body. Namely, if $\vec{\xi}_m$, $\vec{\xi}_f$, and $\vec{\xi}_s$ denote the coordinates of the barycenters O_m, O_f, and O_s with respect to an inertial reference system whose origin O coincides with the barycenter of the whole body, we have

$$\begin{aligned} & m_m\vec{\xi}_m + m_f\vec{\xi}_f + m_s\vec{\xi}_s = \vec{0}, \\ & m_f\vec{\xi}_f = m_f\vec{\xi}_m + \frac{\rho_f}{\rho_s} m_s\vec{\xi}_m - \frac{\rho_f}{\rho_s} m_s\vec{\xi}_s. \end{aligned} \tag{3}$$

In addition, since the motion of the fluid is only due to that of the solid layers, and it is assumed that the fluid is perfect, the velocity field of the fluid has the form (Lamb 1963, art. 136)

$$\vec{v}_f\left(\vec{r},t\right) = \vec{\nabla}\phi, \tag{4}$$

where ϕ is a solution of the Laplace equation in the fluid domain D_f

$$\vec{\nabla}^2\phi = 0. \tag{5}$$

This solution must also fulfill the following boundary conditions on the mantle–fluid surface, ∂D_m, and solid inner core–fluid surface, ∂D_s,

$$\vec{v}_f \cdot \vec{n} = \frac{d\vec{\xi}_m}{dt} \cdot \vec{n} \text{ at } \partial D_m,\ \vec{v}_f \cdot \vec{n} = \frac{d\vec{\xi}_s}{dt} \cdot \vec{n} \text{ at } \partial D_s, \tag{6}$$

the normal vector $\vec{n}$ directed outside the fluid.

A convenient choice of generalized coordinates is to take the position of the solid inner core barycenter with respect to the mantle, that is to say, the three components of $\vec{\eta}_s = \vec{\xi}_s - \vec{\xi}_m$, and the corresponding components of the velocity $d\vec{\eta}_s/dt$.

Accordingly to the previous paragraph the dynamics of the system can be described by giving the temporal evolution of $\vec{\eta}_s$. To this end, we have to construct the expressions of the kinetic and potential energies of the system. Although our framework allows the consideration of non–linear terms, we restrict ourselves to the development of a linear theory as it is customary done in this kind of studies (e.g. Busse 1974, Grinfeld & Wisdom 2005). It means that we are assuming that the instantaneous configuration of the system only departs slightly from the reference configuration. In this way, our study falls in the scope of the theory of small oscillations (e.g. Landau & Lifshitz 2000, sec. 23), so the kinetic and potential energies will be quadratic forms in the generalized velocities and coordinates, respectively.

The kinetic energy of the system is given by the sum of the kinetic energy of each constituent. Specifically, we have that

$$\begin{aligned} \mathcal{T}_m = &\ \frac{1}{2}m_m\left(\frac{d\vec{\xi}_m}{dt}\right)^2,\ \mathcal{T}_s = \frac{1}{2}m_s\left(\frac{d\vec{\xi}_s}{dt}\right)^2, \\ \mathcal{T}_f = &\ \frac{1}{2}\rho_f\int_{D_f}\left[\vec{v}_f\left(\vec{r},t\right)\right]^2\, d\tau^3. \end{aligned} \tag{7}$$

As regards to the kinetic energy of the solid constituents, Eq. (3) allows us to write

$$\vec{\xi}_m = \frac{m_s}{m}\left(\frac{\rho_f}{\rho_s} - 1\right)\vec{\eta}_s = \alpha\vec{\eta}_s,\ \ \vec{\xi}_s = \left(1+\alpha\right)\vec{\eta}_s. \tag{8}$$

So, we obtain

$$\mathcal{T}_m = \frac{1}{2} m_m \alpha^2 \left(\frac{d\vec{\eta}_s}{dt} \right)^2, \, \mathcal{T}_s = \frac{1}{2} m_s \left(1+\alpha\right)^2 \left(\frac{d\vec{\eta}_s}{dt} \right)^2. \tag{9}$$

With respect to the kinetic energy of the fluid and considering Eqs. (5), (6), (7), the Gauss–Ostrogradski theorem provides

$$\begin{aligned} \mathcal{T}_f &= \frac{1}{2}\rho_f \int_{\partial D_m} \phi \left(\vec{\nabla}\phi \cdot \vec{n} \right) dS + \frac{1}{2}\rho_f \int_{\partial D_s} \phi \left(\vec{\nabla}\phi \cdot \vec{n} \right) dS = \\ &= \frac{1}{2}\rho_f \int_{\partial D_m} \phi \left(\frac{d\vec{\xi}_m}{dt} \cdot \vec{n} \right) dS + \frac{1}{2}\rho_f \int_{\partial D_s} \phi \left(\frac{d\vec{\xi}_s}{dt} \cdot \vec{n} \right) dS. \end{aligned} \tag{10}$$

In the small oscillations approximation it can be shown (Escapa & Fukushima 2010) that these integrals can be approached as

$$\int_{\partial D_{m,s}} \phi \left(\frac{d\vec{\xi}_{m,s}}{dt} \cdot \vec{n} \right) dS \simeq \int_{\partial D^{(0)}_{m,s}} \phi^{(0)} \left(\frac{d\vec{\xi}_{m,s}}{dt} \cdot \vec{n} \right) dS, \tag{11}$$

where the superscript (0) denotes quantities evaluated in the reference configuration. The function $\phi^{(0)}$ is given by (Escapa & Fukushima 2010)

$$\phi^{(0)} = \sum_{m=0}^{1} \left[\left(a_{1m} r + \frac{b_{1m}}{r^2} \right) C_{1m} + \left(c_{1m} r + \frac{d_{1m}}{r^2} \right) S_{1m} \right], \tag{12}$$

where C_{1m} and S_{1m}, depending on the spherical coordinates θ and ϕ, denote the real spherical surface harmonics of the first degree (e.g. Escapa & Fukushima 2010), and the coefficients a_{1m}, b_{1m}, c_{1m}, and d_{1m} are linear functions of the components of $d\vec{\eta}_s/dt$.

In this way, taking into account that the surfaces $\partial D^{(0)}_{m,s}$ are spheres centered at O of radius d_f and d_s, the kinetic energy of the fluid turns out to be

$$\mathcal{T}_f = \frac{1}{2} \left(\frac{2}{3} \pi \rho_f c \right) \left(\frac{d\vec{\eta}_s}{dt} \right)^2, \tag{13}$$

with

$$c = 2\alpha^2 d_f^3 - 2\left(1+\alpha\right)^2 d_s^3 + 3\frac{d_f^3 d_s^3}{d_f^3 - d_s^3}. \tag{14}$$

The potential energy of the system V arises from the gravitational interaction among the constituents of the body. However, since the mantle is an spherical shell of constant density, the gravitational potential in its interior is constant (e.g. MacMillan 1958, sec. 29), so it does not influence the equations of motion. Hence, the relevant part of the potential energy only comes from the fluid–solid inner core gravitational interaction. Although it is possible to compute analytically this function, we can derive its form from a physical way of reasoning (Escapa & Fukushima 2010).

Namely, the linearity of the gravitational potential allows us to visualize the fluid–solid inner core subsystem as composed by two homogeneous spheres: one of center O, radius d_f, and density ρ_f, and the other one with center O_s, radius d_s, and density $\rho_s - \rho_f$. By doing so, to construct the gravitational potential energy of these bodies we can substitute the sphere of radius d_s by a material point P with the same mass and located at O_s. So, the gravitational potential energy is equal to the product of the potential of the homogeneous sphere of radius d_f at the position $\vec{\eta}_s$ by the mass of the material point P.

Considering the expression of the potential of a homogeneous sphere (e.g. MacMillan 1958, sec. 29), we find that the part of the gravitational potential energy that influences the dynamics is given by

$$\mathcal{V} = \frac{2\pi}{3} G\rho_f \, (\vec{\eta}_s)^2 \left[\frac{4}{3}\pi d_s^3 \, (\rho_s - \rho_f)\right] = \frac{2\pi}{3} G\rho_f \, m_s \left(1 - \frac{\rho_f}{\rho_s}\right) (\vec{\eta}_s)^2 \,, \quad (15)$$

where G is the universal gravitational constant. This formula coincides with the given by Slichter (1961) and has also been recovered from pure mathematical computations in Escapa & Fukushima (2010).

Therefore, summing up Eqs. (9), (13), and (15) the Lagrangian of the system is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left[m_m \alpha^2 + m_s \, (1+\alpha)^2 + \frac{3}{2}\pi\rho_f c\right] \left(\frac{d\vec{\eta}_s}{dt}\right)^2 - \\ &- \frac{1}{2}\left[\frac{4\pi}{3} G\rho_f \, m_s \left(1 - \frac{\rho_f}{\rho_s}\right)\right] (\vec{\eta}_s)^2 \,. \end{aligned} \quad (16)$$

As it can be seen, the dynamics has been reduced to that of a harmonic oscillator with frequency

$$\omega^2 = \frac{4\pi}{3} G\rho_f \left(1 - \frac{\rho_f}{\rho_s}\right) \frac{m_s}{m_m \alpha^2 + m_s \, (1+\alpha)^2 + \dfrac{2}{3}\pi\rho_f c}. \quad (17)$$

So, the evolution of $\vec{\eta}_s$ is given by

$$\vec{\eta}_s \, (t) = \vec{\eta}_s \, (t_0) \cos\left[\omega \, (t - t_0)\right] + \frac{(d\vec{\eta}_s/dt) \, (t_0)}{\omega} \sin\left[\omega \, (t - t_0)\right], \quad (18)$$

the constants $\vec{\eta}_s \, (t_0)$ and $(d\vec{\eta}_s/dt) \, (t_0)$ providing the initial conditions.

This equation allows us to derive the motion of all the constituents of our three layer Earth model. The equation (17) agrees with that obtained by Grinfeld & Wisdom (2005), in spite of the different method employed in that investigation.

3. Applications

Next we will show some of the possible applications of the previous analytical model to the internal translational motions of different celestial bodies. A detailed analysis is presented in Escapa & Fukushima (2010). Specifically, in this note we will only focus on the influence of the mantle motion in the value of the frequency ω for the Earth and Mercury. Besides, we will examine the dependence of this frequency on the thickness of the external ice I layer of some possible models of icy bodies.

With this aim and in order to determine clearly the influence of the rheological characteristics of the model in the oscillation frequency, it is convenient to re–write Eq. (17), or alternatively its associated period. It can be achieved by considering different asymptotic situations of the model (Escapa & Fukushima 2010). In particular, after comparing the period of our model with that of a sphere moving in an unbounded fluid (no mantle case), and in a fluid limited by a external concentric spherical shell to the inner core (mantle in rest case), we can write

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{3\pi}{G} \frac{\left[1 + \frac{1}{2}\,\frac{\rho_f}{\rho_s} + \frac{3}{2}\left(\frac{\rho_f}{\rho_s}\right)^2 \frac{m_s}{m_f} + \Delta_m\right]}{\rho_f\left(1 - \frac{\rho_f}{\rho_s}\right)}}, \tag{19}$$

with

$$\Delta_m = -\,\frac{m_s}{m}\left(1 - \frac{\rho_f}{\rho_s}\right)^2,\ m = m_m + m_f + m_s. \tag{20}$$

The second term inside the bracket of the numerator in Eq. (19) gives the contribution arising from the motion in an unbounded fluid, the third one the correction introduced when considering an external concentric spherical shell, and the last one, Δ_m, accounts for the contribution of the mantle motion. This expression shows clearly the influence of each constituent on the dynamics, since the period is a relatively simple function ρ_f, and the ratios ρ_f/ρ_s, m_s/m_f and m_s/m.

From this expression we will be able to obtain the relative difference between the associated periods of the translational motion when the mantle can move T; or when it is assumed to be in rest $\hat{T}$, that is to say, by taking $\Delta_m = 0$ in Eq. (19). This difference is computed from

$$\Delta T = \frac{T - \hat{T}}{T}. \tag{21}$$

If, in addition, it is assumed that Δ_m is small with respect to the other terms appearing in Eq. (19), as it is the case for the Earth and Mercury, we can derive a simple first order analytical expression that accounts for that relative difference ΔT. Namely, by performing a Maclaurin's expansion of Eq. (21) in Δ_m, we found

$$\Delta T = \frac{1}{2}\frac{\Delta_m}{1+\frac{1}{2}\,\frac{\rho_f}{\rho_s}+\frac{3}{2}\left(\frac{\rho_f}{\rho_s}\right)^2\,\frac{m_s}{m_f}} + \ldots \tag{22}$$

This expression in combination with Eq. (19) allows to understand why the effect of the translational motion of the mantle is almost irrelevant for the Earth and Mercury. First, let us note that ΔT is negative since it is also the case of Δ_m. As a result, the motion of the mantle shortens the period of the oscillations. This is evident from Eq. (19): since $\Delta_m < 0$, when considering the mantle contribution the numerator is smaller, hence the period is shorter.

With regard to its magnitude, the relative difference is proportional to Δ_m what means that depends directly on the factor m_s/m and the contrast of densities (Eq. 20). In addition, the approximation of ΔT is divided by a function that contains, not only the contrast of densities ρ_f/ρ_s, but also the ratio between the mass of the inner core and the mass of the fluid. Hence, the effect of a massive inner core in Δ_m may be compensated in some amount when divided by m_s/m_f, since if the mass of the inner core is significant in comparison with that of the whole body, it will also be significant in comparison with the mass of the fluid, so the ratio m_s/m_f will have a large value.

Table 1. Influence on the mantle motion in the internal translations period of the Earth and Mercury

Body	m_s/m_f	ρ_f (kg m^{-3})	ρ_f/ρ_s	Δ_m	T (h)	$\widehat{T}$ (h)
Earth	0.05	12000	0.92	-0.0000	4.2356	4.2358
Mercury	6.35	8000	0.84	-0.0149	8.3900	8.3977

In Table 1 we show the different quantities entering in Eq. (19) and the associated periods T and $\hat{T}$ for Earth and Mercury, considering the values of the physical characteristics of the Earth and Mercury models given in Grinfeld & Wisdom (2005). We see that in the Earth case the influence of the mantle motion is negligible as a consequence of the smallness of the inner core and the similarity of the densities ρ_f and ρ_s. It implies that Δ_m is very small and, therefore, the relative percentage variation ΔT is minute.

For Mercury the variation ΔT is almost insignificant, but the reasons are different. In contrast to the Earth the ratio m_s/m accounts for about

60% of the mass of Mercury. On the other hand, the density contrast of Mercury is, roughly speaking, low as in the Earth case. These two facts explain the origin of the value Δ_m which remains relatively small due to the dependence with $(1-\rho_f/\rho_s)^2 \sim 0.025$. At the same time, and it is also an important difference with the Earth case, since the inner core is massive the ratio m_s/m_f is also larger. This ratio is multiplied by the factor $(\rho_f/\rho_s)^2$, but it does not change significantly its value due to the fact that ρ_f is close to ρ_s. So, when divided by the terms containing m_s/m_f the contribution of the mantle Δ_m motion is reduced even more, providing a percentage relative variation of $\Delta T \simeq -0.1\%$.

Therefore, the influence of the mantle motion on the translational period of our model does not only depend on the relative size of the inner core, but also on the density contrast between the fluid and the inner core and the ratio of their respective masses. Hence, a significative influence of the mantle motion requires, in addition to a massive core, a non too small fluid layer with a high density contrast. In this situation, the influence of the ratio m_s/m_f can be relatively compensated by the factor $(\rho_f/\rho_s)^2$ and the contribution Δ_m is not substantially reduced .

The second application that we are going to treat is concerned with some icy bodies that also present a three layer structure but composed of an external ice I layer, covering a subsurface water–ammonia ocean that contains a large rocky inner core. This kind of models have been worked out in Hussmann et al. (2006). They who found that, under the proper conditions, some medium–sized outer planet satellites and large trans–neptunian objects might have a subsurface ocean due to the presence of ammonia. In particular, we will consider the three layer models of Europa, Titania, Oberon, Triton, and Pluto developed in that work.

Depending on the initial concentration of ammonia the depth of the subsurface ocean of the model is different. In turn, it implies a change on the thickness of the ice I layer. In this way, for each body we have a family of models differing in the thickness of the ice I layer that runs into an interval of possible values $[d_{\min}, d_{\max}]$. From our perspective, it will be interesting to analyze if the period of the internal translations (Eq. 19) could differentiate those models, that is to say, to determine whether the period depends significatively on the thickness of ice I layer or not. If it is the case, observing the value of the period of the internal translation could provide another indirect mean to constrain the internal structure of these bodies.

In Table 2, and considering the data provided by Hussmann et al. (2006), we have computed the different periods associated to the internal translational motions of some three layer models of Europa, Titania, Oberon, Triton, and Pluto. Since the densities of the ice and liquid layers were taken to be equal, and the sum of the depths of these layers was constant, the sum of the masses $m_m + m_f$ remains also constant for each body too (see Hussmann et al. 2006). Let us note that for these bodies

Table 2. Physical parameters for some models of icy bodies with a possible subsurface ocean and their associated internal translational periods

Body	ρ_f/ρ_s	d_s (km)	m_s/m	d (km)	m_s/m_f	T (h)
Europa	0.24	1405.0	0.92	79.5	22.58	6.39
				77.5	22.01	6.32
				74.8	21.27	6.24
				70.0	20.07	6.11
				57.0	17.38	5.79
Titania	0.29	519.8	0.58	253.1	36.76	9.03
				229.7	14.28	6.29
				217.6	10.68	5.73
Oberon	0.29	481.0	0.54	264.4	33.93	8.75
				241.1	13.17	6.15
Triton	0.29	1017.0	0.72	200.5	7.66	5.11
				194.9	7.32	5.05
				187.5	6.91	4.97
				174.8	6.29	4.86
				143.9	5.13	4.63
Pluto	0.29	830.2	0.64	260.6	8.22	5.27
				248.7	7.28	5.11
				234.9	6.40	4.94
				214.5	5.41	4.75
				179.9	4.23	4.51

the contrast between the inner core and the fluid densities is very high, since they are composed of very different materials: silicate rock versus H_2O (ρ_f =1000 kg m^{-3}). In particular, we have that ρ_f/ρ_s is about 0.29 (0.24 in the case of Europe), in contrast to the Earth or Mercury situations where we had the values 0.92 and 0.84, respectively. In addition, all these bodies have large rocky core when compare with the total mass of the body. In this sense, they are in the opposite situation as the Earth case: they have large inner cores and a high contrast density.

From Table 2 it is derived that for all the bodies the value of the internal translational period lies between four and nine hours, this value differentiating among the diverse possible models for the same body. To discuss this differentiation, i.e. the dependence of the period with the thickness of the ice I layer, or alternatively, with the depth of the subsurface ocean, in Figure 2 we have plotted the period of these bodies against a normalized–like thickness of the ice I layer $\bar{d}$, instead of the real thickness

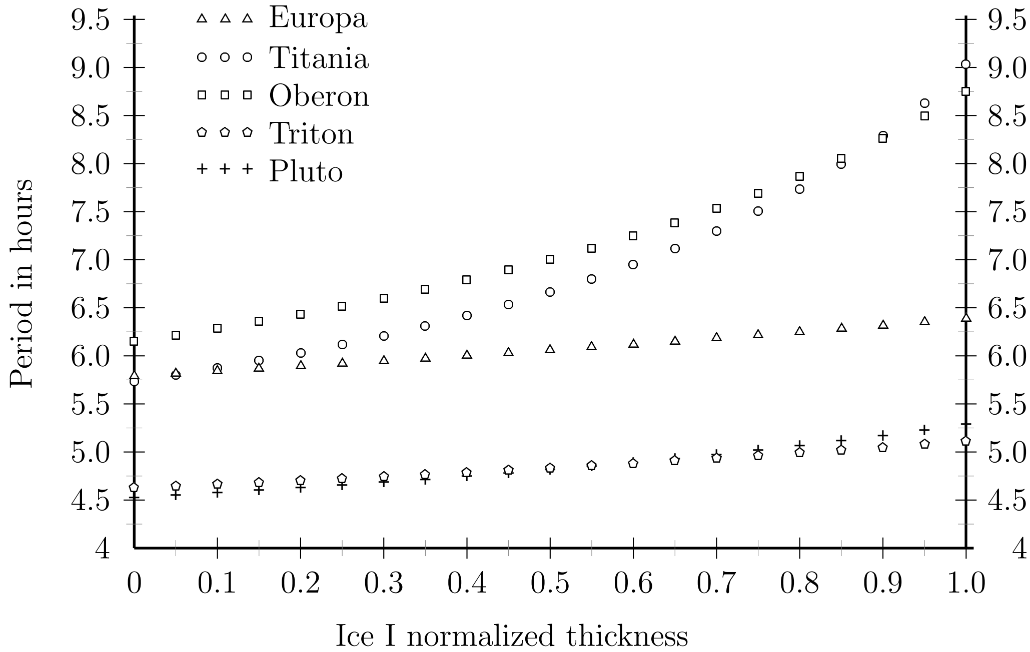

Figure 2. Dependence of the internal translational period with the thickness of the ice I layer for different models of Europa, Titania, Oberon, Triton, and Pluto

d. We have introduced this variable $\bar{d}$ to represent in one single graphic all the situations and is defined as

$$\bar{d} = \frac{d - d_{\min}}{d_{\max} - d_{\min}}, \;\; \bar{d} \in [0, 1] \,, \tag{23}$$

where $d_{\min}$ and $d_{\max}$ denote, respectively, the minimum and maximum thickness of the ice I layer of the body. For example, in the case of Europe $d_{\min} = 57.0$ km and $d_{\max} = 79.5$ km which correspond to the normalized values $\bar{d} = 0$ and $\bar{d} = 1$.

Table 2 and Figure 2 reflect that the period increases with the increase of the thickness of the external ice layer. As a matter of fact, with our approximations the period of each body is exclusively a function of $\bar{d}$. This circumstance comes from Eq. (19): the dependence of T on $\bar{d}$ arises only through the dependence of T on m_f, since the parameters m, m_s, d_m, ρ_f and ρ_s are constant for all the models of the same body. In addition, the constancy of $m_m + m_f$ and ρ_m implies that the mass of the ice layer is only a function of $\bar{d}$. Namely, we have that

$$m_m = \frac{4}{3}\pi\rho_m \left(d_m^3 - d_f^3\right) = \frac{4}{3}\pi\rho_m \left[d_m^3 - (d_m - d)^3\right] . \tag{24}$$

On the other hand, the expression of the period given by Eq. (19) can be easily written in terms of m_m what provides the dependence $T = T\left(\bar{d}\right)$. Since when the ice thickness increases the mass of the mantle m_m increases too, the denominator $m - m_s - m_m$, equal to m_f appearing in Eq. (19) decreases (recall that m and m_s take the same values for all the models of each body). So, the fraction m_s/m_f is greater and it is also the case for the numerator of the fraction inside the square root. Therefore, the period is increased.

Let us underline that the difference in the value of the periods associated to models with different ice-I thickness is significative. In fact, the relative difference between the opposite situations $\bar{d} = 0$ and $\bar{d} = 1$ for each body, $1 - T(0)/T(1)$, is about a 10%, with the exceptions of Titania and Oberon when it reaches a 37% and a 30%, respectively. In addition in these cases the absolute differences are specially relevant, since they are of the order of three and two hours.

Another interesting feature of these icy bodies (Escapa & Fukushima 2010) is that the influence of the motion of the ice I layer on the value of the period of the translational oscillation cannot be neglected, since ignoring that contribution can cause relative error of the order of a 10% in the period of oscillation. It is an important difference with respect to the Earth and Mercury cases, showing the need of considering in the modeling of the internal dynamics of these systems the motion of the external layer.

4. Summary

In this investigation we have presented an analytical model to describe the internal translational motion of a three layer body. This theoretical treatment has been developed from the methods of dynamical systems. In particular, the differential equations that governs the time evolution of the constituents have been obtained from a Lagrangian approach by constructing the kinetic and gravitational potential energy of the system. This approach avoids the computation of the interactions among the fluid and the solids, since they are automatically included in the kinetic energy of the system. In a linear approximation, the equations of motion have led to a solution of an oscillatory type. Our analysis has allowed us to express the frequency of that oscillation in terms of some basic physical characteristic of the model, providing a clear interpretation of the influence of the body constituents on the value of the frequency.

As two initial applications of our theory we have considered the internal translation motions of the Earth and Mercury, and of some possible models of the icy bodies Europa, Titania, Oberon, Triton, and Pluto. In the Earth and Mercury cases we have found the reason that explains why the motion of the mantle does not influence significatively the value of the translational frequency. For the Earth it is due to the small inner core, whereas for Mercury, in spite of the fact of having a large inner core, it is the joint consequence of a small density contrast, and a thin fluid out core when compared with the mass of the solid inner core.

As regard to the icy bodies we have shown that the translational frequency could provide a possible method to constrain the internal structure of these bodies, since the differences in its value for different models of the same body can be of the order of hours. In this way, we could infer some limit to the thickness of the ice I layer, or the depth of the subsurface oceans, of these bodies. This could have some interest in the design of the Europa Jupiter System Mission (2009), one of whose objectives is to determine the internal structure of Europa and Ganymede.

Finally, let us point out that one of the advantages of our dynamical system approach is that the extension within this framework seems reliable. So, in addition to incorporate some improvements in the physical characteristics of the model, it would also be possible to include in our study the internal rotational motions drawing a clear picture of the roto–translatory coupling of the internal rigid motions. This work is now in progress and will be presented in a forthcoming communication.

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