Dynamics of gas sphere under self-gravity

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Abstract

A new dynamical solution for a gas sphere under self-gravity is presented to describe a development of a gas sphere from a motion-less state to a state of expansion with a constant speed and a reflection phenomenon in the dynamics of the surface of the sphere.

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1 Introduction

The polytropic model of stellar structure is often introduced in studies of the different stages of stellar evolution. The equation governing the static equilibrium of a polytropic gas sphere is called the Emden equation, whose solution representing the structure of star is known as the Emden solution [1]. Quasi-statical change of stellar structure expressed by the Emden solution is believed to describe the evolution of star when the Emden solution is dynamically stable. The dynamical stability is not guaranteed unless the polytropic index N is less than 3 [2]. A gas sphere of $N = 3$ is borderline to the dynamical stability and is also interesting since the polytropic index 3 corresponding to the adiabatic constant $\gamma = 4/3$. There are some interesting stellar gases with $\gamma = 4/3$, such as a radiation pressure dominant gas, an extremely relativistic gas and a gas with degenerate electrons [2].

In order to extend the static Emden solution to dynamical states, Munier and Feix derived an explicit self-similar solution which describes a decelerating expansion of a gas sphere of $N = 3$ [3]. Recently, another explicit self-similar solution has been obtained to describe a non accelerating nor decelerating expansion (or contraction) of a gas sphere of $N = 3$ [3].

In this paper,we present a new dynamical solution for a gas sphere of $N = 3$, which is not self similar in general but includes the three known Emden-type solutions described above as special cases. The new dynamical solution describes such interesting phenomena as a development of a gas sphere from a motion-less state to a state of expansion with a constant speed and a reflection phenomenon in the dynamics of the surface of a sphere.

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2 Dynamical solution of gas sphere

The equations governing the spherically symmetrical flow of a polytropic gas of adiabatic index γ under the influence of its own gravitation are

$$\rho_t + u\rho_r + \rho\left(u_r + \frac{2}{r}u\right) = 0, \tag{1}$$

$$u_t + uu_r + \rho^{\gamma-2}\rho_r + \sigma_r = 0, \tag{2}$$

$$\sigma_{rr} + \frac{2}{r}\sigma_r = \rho, \tag{3}$$

where the pressure P, density ρ, radial velocity u, gravitational potential σ and radius r are normalized by P_0, ρ_0, $u_0 = (\gamma P_0/\rho_0)^{1/2}$, $\gamma P_0/\rho_0$, r_0, respectively; G is the gravitational constant and $P = \rho^\gamma$. The acoustic time scale r_0/u_0 , where $r_0 = \gamma P_0/(4\pi G\rho_0^2)$, is set to equal to the time scale of free fall $(4\pi G\rho_0)^{-1/2}$. The system of Eqs. (1), (2) and (3) admits a following Lie point symmetry

$$\mathbf{V} \equiv (t+t_0)\partial_t - (\gamma-2)r\partial_r - 2\rho\partial_\rho - (\gamma-1)u\partial_u - \gamma\phi\partial_\phi, \tag{4}$$

where $\phi = \sigma_r$ and t_0 is an arbitrary constant. The symmetry (4) is derived through the standard procedure. Solving the Lie equation associated with the infinitesimal generator (4), we introduce a new independent variable T defined as

$$dT = \frac{dt}{t+t_o}, \tag{5}$$

and scaled variables

$$\rho = \frac{R(x,T)}{(t+t_0)^2}, \quad u = \frac{U(x,T)}{(t+t_0)^{\gamma-1}}, \quad \phi = \frac{\Phi(x,T)}{(t+t_0)^\gamma}, \tag{6}$$

where

$$x = \frac{r}{t^{2-\gamma}}, \quad T = \log(t+t_0).$$

Let us make the following ansatz

$$R(x,T) = \tilde{R}(y)\exp\left\{\int\left(2-3\overline{U}\right)dT\right\}, \tag{7}$$

$$U(x,T) = \overline{U}(T)x, \tag{8}$$

$$\Phi(x,T) = \tilde{\Phi}(y)\exp\left\{\int\left(\gamma-2\overline{U}\right)dT\right\}, \tag{9}$$

where

$$y = x\exp\left\{\int\left(2-\gamma-\overline{U}\right)dT\right\}.$$

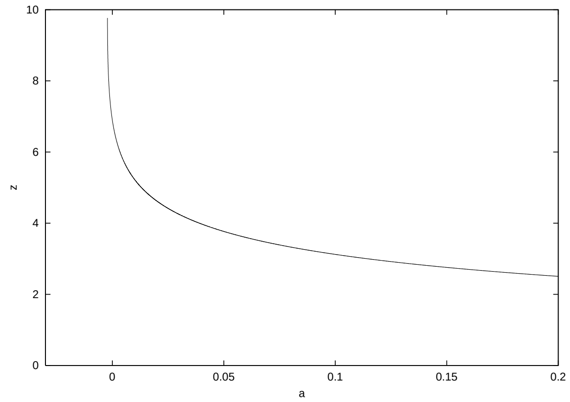

Figure 1: z vs. the acceleration parameter a

This ansatz ensures that Eq. (1) is satisfied automatically. For $\gamma = 4/3$, Eqs. (2) and (3) lead to

$$ay + \tilde{R}^{\gamma-2}\tilde{R}_y + \tilde{\Phi} = 0, \tag{10}$$

$$\overline{U}_T + \overline{U}(\overline{U} - 1) = a\exp\left\{\int\left(2 - 3\overline{U}\right)dT\right\}, \tag{11}$$

$$\tilde{\Phi}_y + \frac{n}{y}\tilde{\Phi} = \tilde{R}, \tag{12}$$

where a is an arbitrary constant. Eqs. (10) and (12) give a modified Emden equation for the density profile $\tilde{R}(y)$

$$\theta_{y'y'} + \frac{2}{y'}\theta_{y'} + \theta^3 + 3a = 0, \quad \theta = \tilde{R}^{1/3}, \quad y' = \frac{y}{\sqrt{3}}. \tag{13}$$

According to numerical integration of Eq.(13) with the boundary condition $\theta(0) = 1, \quad \theta_{y'}(0) = 0$, the density profile has the first zero at finite $y' = z$ for $a_0 > -0.00219$ as shown in Figure. 1. The value of z determines the radius of a gas sphere. For $a = 0$, z takes the Emden's static value $z = 6.89685$. Since the term ay' comes from the inertia term and represents acceleration effect for $a > 0$, we call a as the acceleration parameter in this paper. In the acceleration case, the surface of a gas sphere feels inward inertia force and shrinks from the Emden's static value. For $a < 0$, the surface feels outward inertia force and expands. At $a \approx -0.00219$, z diverges to infinity and a localized gas sphere disappears.

It is easy to see that Eq.(11) takes the following integrable form

$$f_{tt} = \frac{a}{f^2}, \tag{14}$$

where

$$f(t) = \exp\left(\int^T UdT\right). \tag{15}$$

Eq. (14) can be integrated once and yields

$$\frac{1}{2}f_t^2 + \frac{a}{f} = b, \tag{16}$$

where b is an integration constant. In terms of $f(t)$, the solution (6) is rewritten as

$$y = \frac{r}{f}, \quad \rho = \frac{\tilde{R}(y)}{f^{n+1}}, \quad u = f_t y, \quad \phi = \frac{\tilde{\Phi}(y)}{f^n}, \tag{17}$$

where $\tilde{R}(y)$ is expressed by a localized solution of Eq.(13) while Eq.(10) gives $\tilde{\Phi}(y)$ in terms of $\tilde{R}(y)$. It is confirmed that the solution (17) satisfies the boundary condition at the moving surface of sphere. While the surface is moving with a speed $\tilde{u} = \sqrt{3}f_t z$, the flow speed at the surface is given by the same one, i.e. $u = f_t y = \sqrt{3}f_t z$.
When $b = 0$ and $a < 0$, Eq.(16) yields

$$f(t) = \left(\frac{3\sqrt{-2a}}{2}(t - t_0)\right)^{2/3}$$

and the solution (17) becomes identical with the known self-similar solution [3]. In the absence of the inertia term (i.e. $a = 0$), Eq.(13) reduces to the Emden equation. For $b = 0$ and $a = 0$, $f(t)$ is constant and Eq.(17) gives the static Emden solution. For $b > 0$ and $a = 0$, Eq.(16) gives $f(t) = \sqrt{b}t$ and Eq.(17) reproduces the expanding Emden solution which is expanding with the constant acoustic speed [5]. For the other values of a and b, a new class of expanding solution with positive or negative acceleration is obtained. Since $\tilde{u} = \sqrt{3}f_t z$, the acceleration rate of the surface is given by $\tilde{u}_t = a\sqrt{3}z/f^2$, of which sign is the same as the a's sign.

To demonstrate the evolution of the solution (17) , we consider an initial value problem for Eq.(16) such that $f(t)|_{t=0} = 1$ in order that $y|_{t=0} = r$.

First of all, let us consider the case in which $a > 0$ and the acceleration rate of the surface is radially positive. Since $f \to \infty$ as $t \to \infty$, $f_t \to \sqrt{2b}$ asymptotically and the surface eventually expands with a constant speed $\sqrt{6b}z$. In this case, we can take a motionless initial condition so that $f_t|_{t=0} = 0$ or $u|_{t=0} = 0$ by choosing the arbitrary (positive) constant b as $b = a(> 0)$. The expansion speed of the surface ($\tilde{u} = \sqrt{3}zf_t$) is depicted for $a = b = 0.01$ in Fig.2.

Next, we consider the radially decelerating case (i.e. $a < 0$), where we take $f_t|_{t=0} > 0$ as an initial condition so that the initial velocity of the surface is positive. Then, the initially expanding surface gradually decelerates due to the negative inertia term. In this case, Eq.(16) reads

$$\frac{1}{2}f_t^2 = b + \frac{|a|}{f}. \tag{18}$$

For $b > 0$, there are no reflection points and the expanding speed of the surface decreases to a constant speed , that is,

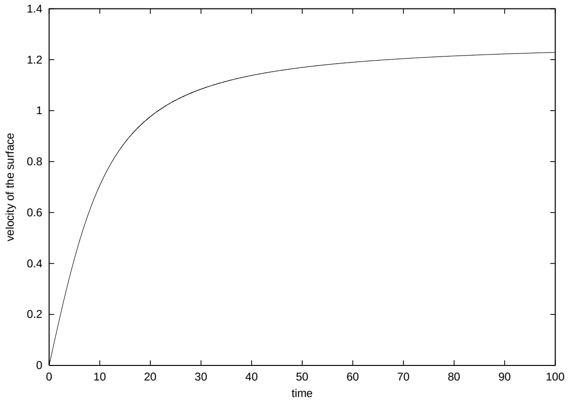

Figure 2: Expansion velocity of the surface for a=b=0.01.

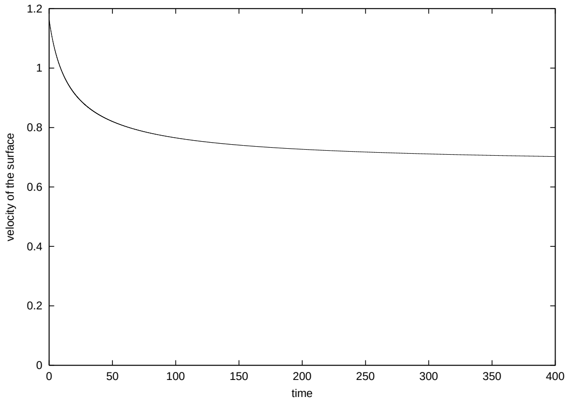

Figure 3: Velocity of the surface for a=-0.002 and b=0.001.

$$\lim_{t\to\infty} \tilde{u} = \sqrt{6b}z, \quad \lim_{t\to\infty} \tilde{u}_t = 0. \tag{19}$$

We illustrate the velocity of the surface for $a = -0.002$ and $b = 0.001$ in Fig. 3. When $b < 0$, a reflection point appears at $f = a/b$. The deceleration of the surface stops at the reflection point and then the gas sphere began to shrinks. The reflection time t_r is given as When $a < 0$ and $b < 0$, the equation (16) yields

$$t_r = \left| \frac{\sqrt{(2b-2a)}}{2b} - \frac{a}{b\sqrt{-2b}} \arctan\sqrt{\frac{2b-2a}{-2b}} \right|.$$

In this reflection case, the velocity of the surface is depicted for $a = -0.002$ and $b = -0.001$ in Fig.4, where $t_r = 57.485$.

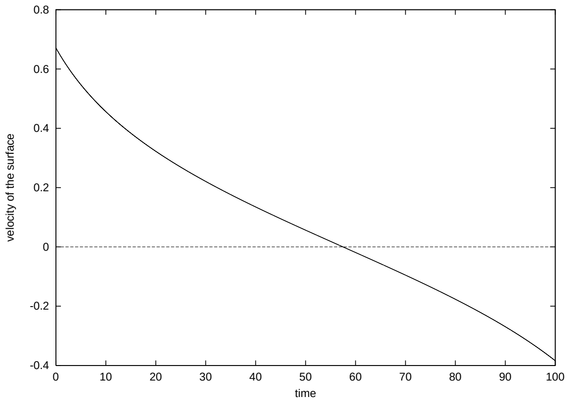

Figure 4: Velocity of the surface for a=-0.002 and b=-0.001.

3 summary

We present a new dynamical solution for a gas sphere under self-gravity, which not only unifies the three Emden-type solutions for the polytropic index $N = 3$ but also describes the following interesting phenomena.
For positive values of acceleration parameter a, the radius of the gas sphere decreases from the Emden's radius due to inward inertia force, while the gas sphere expands for the negative a and the distinct surface disappears when $a < -0.00219$. The new solution also describes acceleration of the surface of a gas sphere from a motion-less state to a state of expansion with a constant speed for the positive a and a reflection phenomenon in the dynamics of the surface of a sphere for the negative a.

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