Synchronised Triangles in the figure-eight solution under 1/r^2 potential

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with
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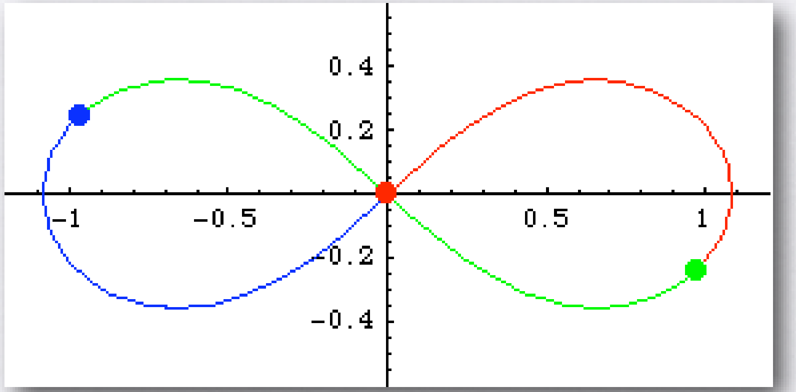
- Three tangents theorem *(with Fukuda and Ozaki)*
- Three-body choreography on the lemniscate
  (with Fukuda and Ozaki)
- Inconstancy of the moment of inertia
  (with Fukuda and Ozaki)
- Convexity of each lobe *(with Montgomery)*

- ***Synchronised triangles in the figure eight solution under 1/r^2 potential.***

Three-Body Figure-Eight Choreography

- C. Moore (1993): finds numerically
- A. Chenciner and
  R. Montgomery (2000):
  prove the existence



- C. Simó (2000):
  finds lots of N-body choreography numerically

Three-Body Figure-Eight Choreography

$$i = 1, 2, 3, \ m_i = 1$$

$$\ddot{q}_i = \sum_{j \neq i} \frac{q_j - q_i}{|q_j - q_i|^3},$$

$$\begin{cases} q_1(t) = q(t), \\ q_2(t) = q(t + T/3), \\ q_3(t) = q(t + 2T/3), \end{cases}$$

$$\sum_i q_i = 0, \ \sum_i q_i \wedge \dot{q}_i = 0.$$

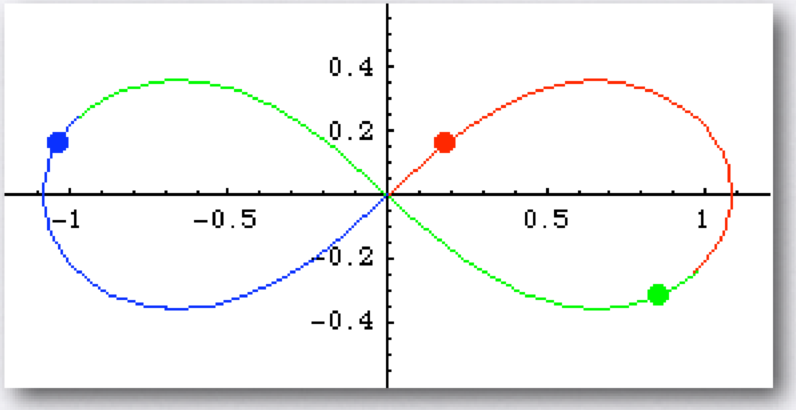


Figure-Eight solution for V_α

$$V_\alpha = \begin{cases} \alpha^{-1} r^\alpha \text{ for } \alpha \neq 0 \\ \log r \text{ for } \alpha = 0 \end{cases}$$

Numerical evidence
Moore: Exist for $\alpha < 2$
CGMS: Exist for $\alpha < 0$ and Stable $\alpha = -1 \pm \epsilon$

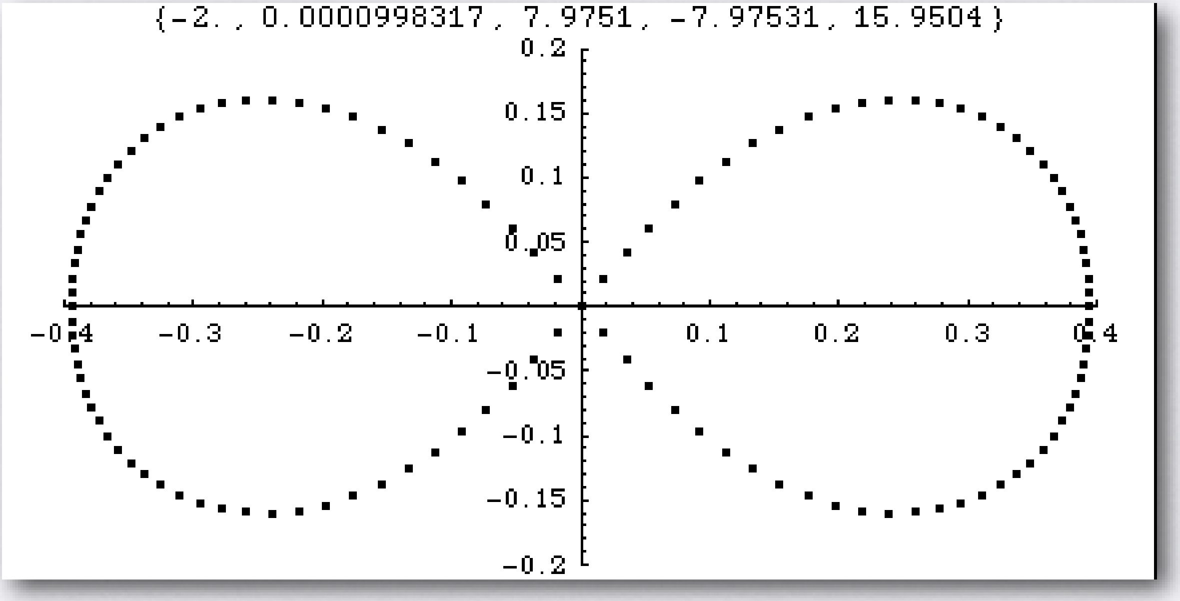


Figure-Eight for $-2 \leq \alpha \leq 1$, $T = 1$.

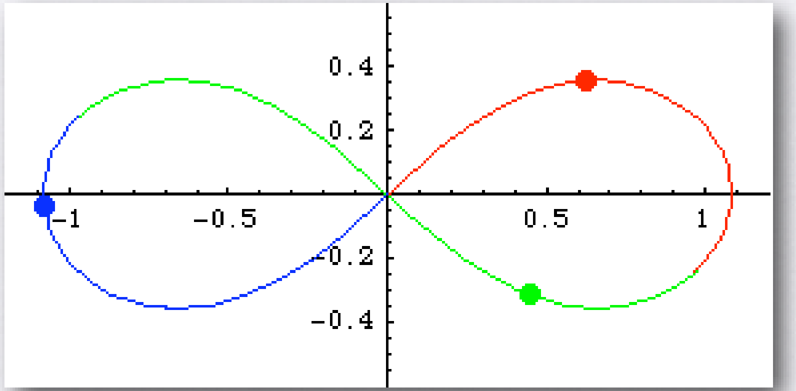
Figure Eight has Zero Angular momentum

Why $L = 0$?

Total angular momentum is conserved.
Therefore,

$$\sum_i q_i \wedge \dot{q}_i = \sum_i < q_i \wedge \dot{q}_i >= 0.$$

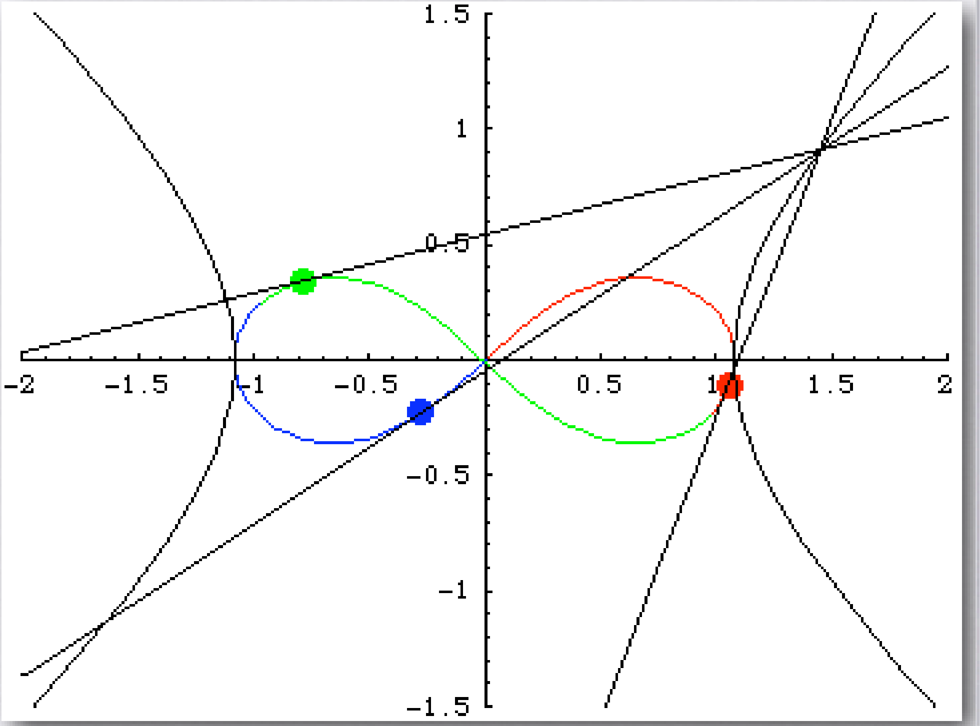
<•> : time average



Then, what does $L = 0$ mean?

Three Tangents Theorem (FFO)

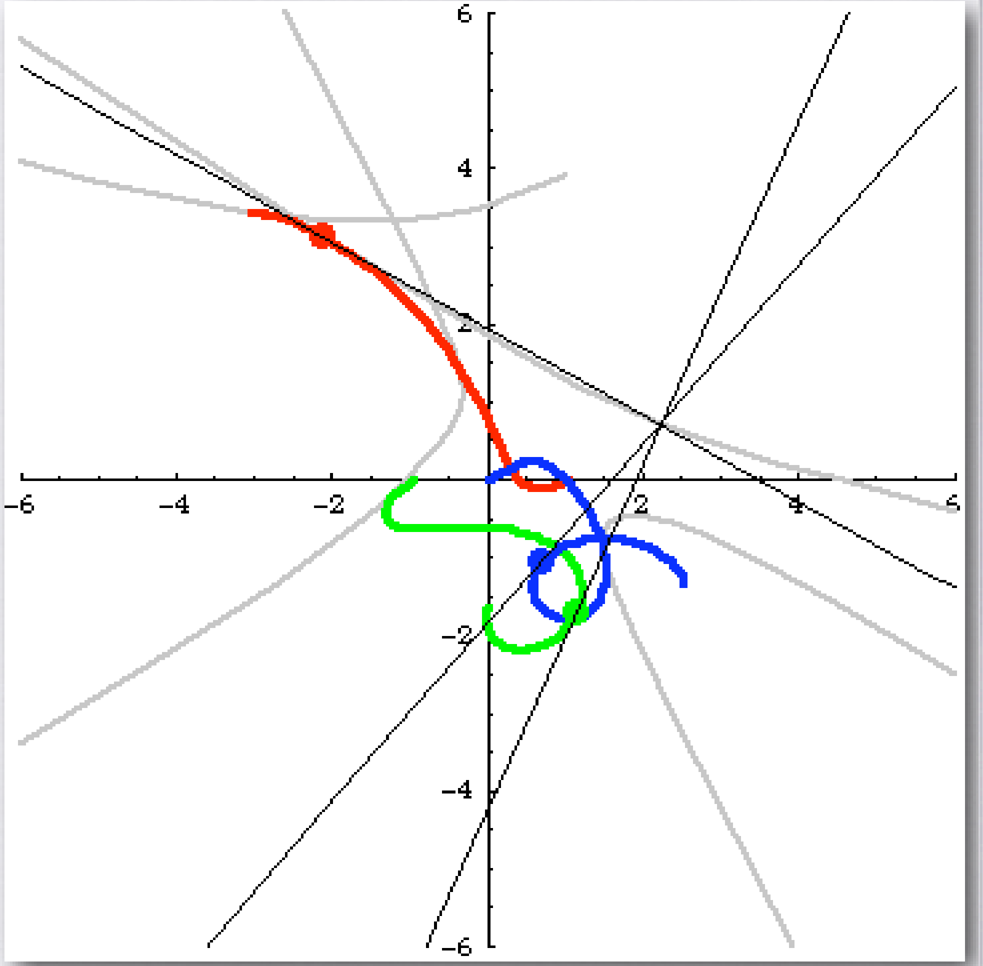
Theorem (Three Tangents). *If* $\sum_i p_i = 0$ *and* $\sum_i q_i \wedge p_i = 0$, *then three tangents meet at a point.*



Three Tangents Theorem

Theorem (Three Tangents). *If* $\sum_i p_i = 0$ *and* $\sum_i q_i \wedge p_i = 0$, *then three tangents meet at a point.*

holds for general masses m_i.



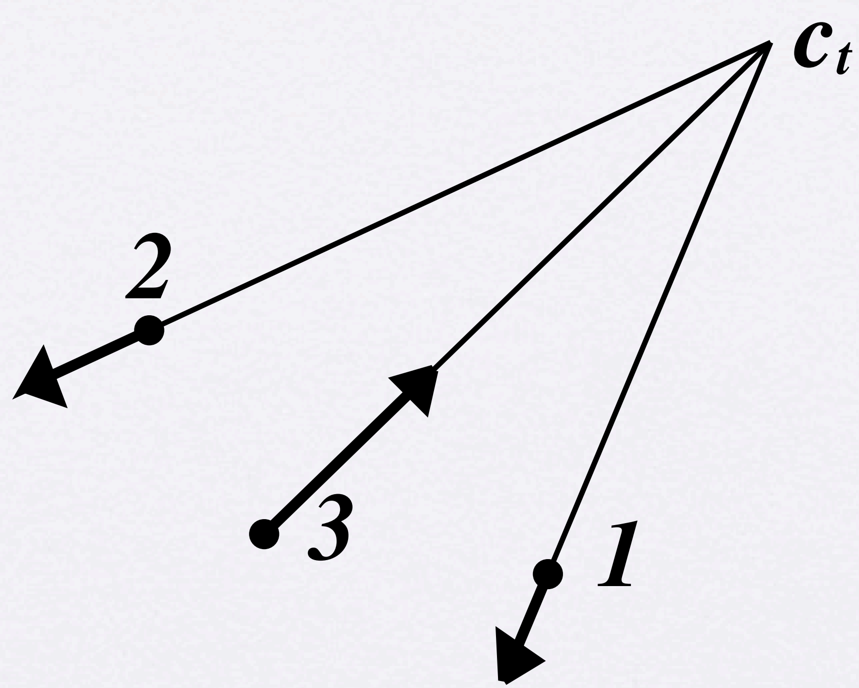
$m_1 = 1.0, m_2 = 1.1, m_3 = 1.2$

Three Tangents Theorem

Theorem (Three Tangents). *If* $\sum_i p_i = 0$ *and* $\sum_i q_i \wedge p_i = 0$, *then three tangents meet at a point.*

Proof. Let C_t be the crossing point of two tangents p_1 and p_2.

Then, $\sum_i (q_i - C_t) \wedge p_i = 0$,
$(q_1 - C_t) \wedge p_1 = 0$ and
$(q_2 - C_t) \wedge p_2 = 0$.
$\therefore\ (q_3 - C_t) \wedge p_3 = 0$. □

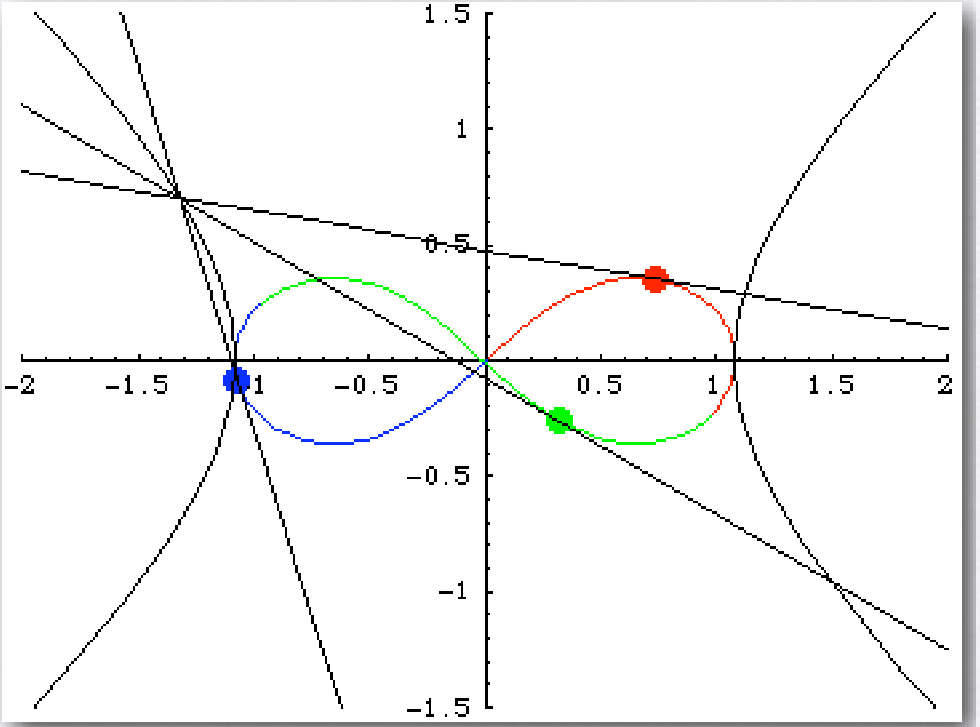


$$C_t = -\frac{(q_i \wedge p_i)p_j - (q_j \wedge p_j)p_i}{p_i \wedge p_j}$$

C_t: the "Center of Tangents"

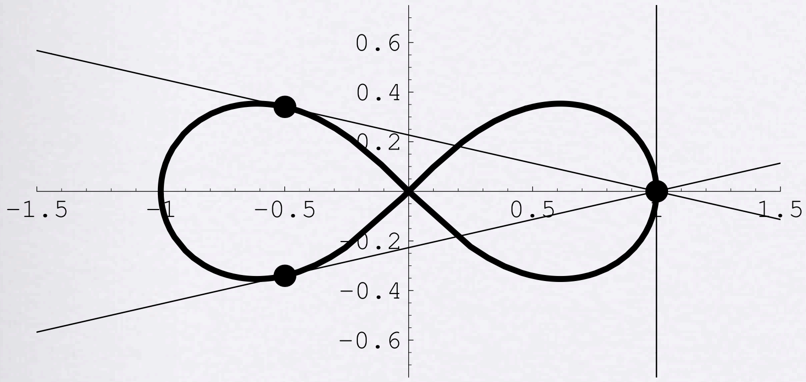
Three Tangents Theorem

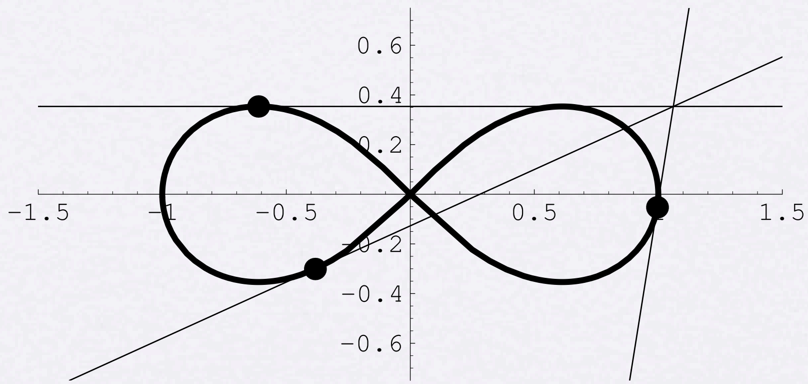
- Shape of the orbit of Figure Eight *x(t)* and the orbit *C(t)* are still unknown.
- Three Tangents Theorem gives a criterion for the orbit.
- For example ...



Simplest Curve: Fourth order polynomial

$$x^4 + \alpha x^2 y^2 + \beta y^4 = x^2 - y^2$$





$$\alpha = 2$$

$$\beta = 1$$

(numerical)

Candidate:
Lemniscate
and its scale transform

$$(x^2 + y^2)^2 = x^2 - y^2$$
$$x \to \mu x, y \to \nu y$$

Three Body Choreography on the Lemniscate (FFO)

Choreograpgy on the Lemniscate

$$q(t) = \left(\frac{\mathrm{sn}(t)}{1+\mathrm{cn}^2(t)}, \frac{\mathrm{sn}(t)\mathrm{cn}(t)}{1+\mathrm{cn}^2(t)} \right) \text{ with } k^2 = \frac{2+\sqrt{3}}{4},$$

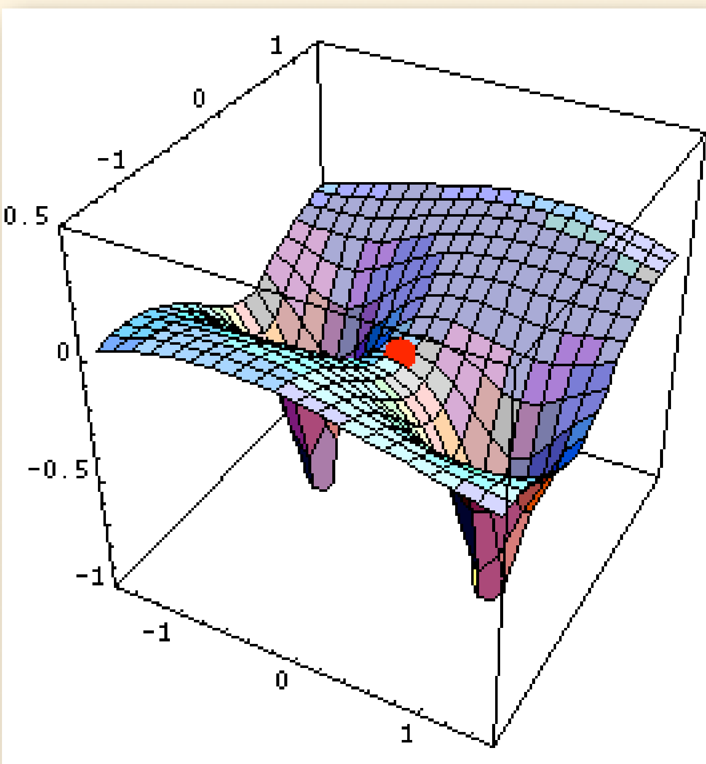
$$\begin{cases} q_1(t) = q(t), \\ q_2(t) = q(t+T/3), \\ q_3(t) = q(t+2T/3), \end{cases}$$

satisfies the equation of motion $\ddot{q}_i = -\frac{\partial}{\partial q_i} U$ with

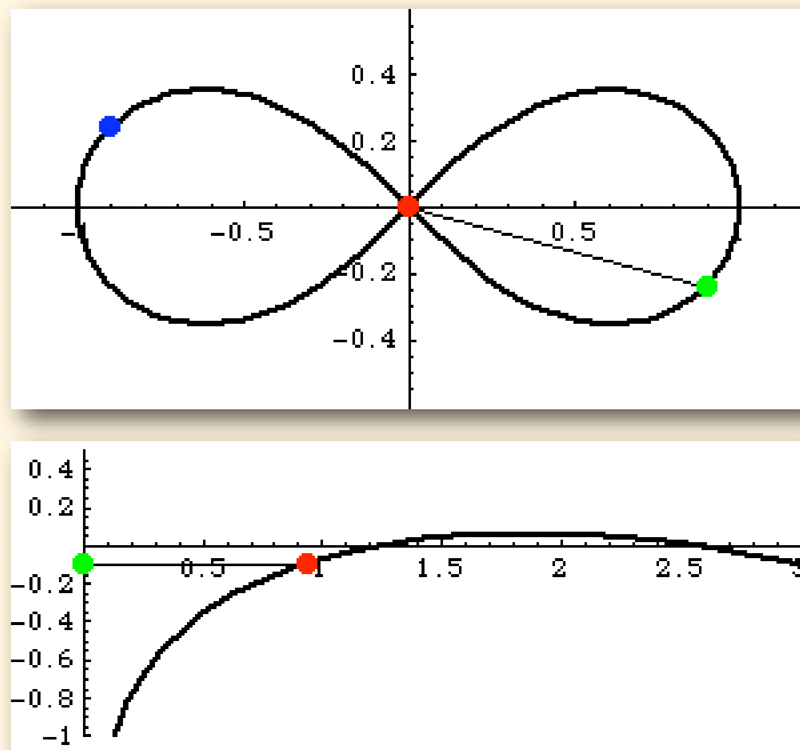
$$U = \sum_{i<j} \left(\frac{1}{2} \ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2 \right).$$

Potential Energy

$$V = \sum_{i<j} V_{ij}, \; V_{ij} = \frac{1}{2} \ln r_{ij} - \frac{\sqrt{3}}{24} r_{ij}^2.$$



$V_{12} + V_{13}$



V_{12}

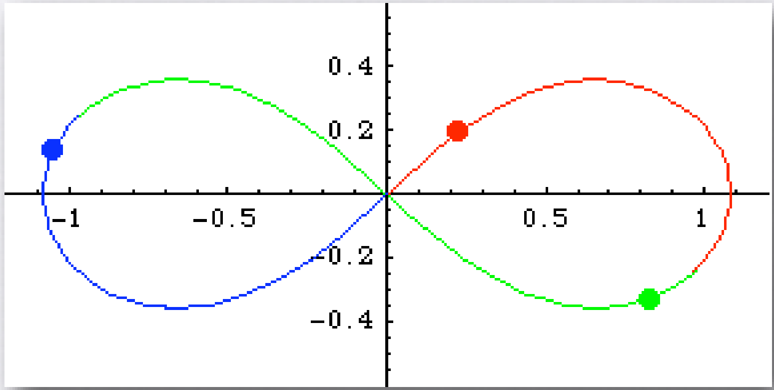
Inconstancy of the Moment of Inertia (FFO)

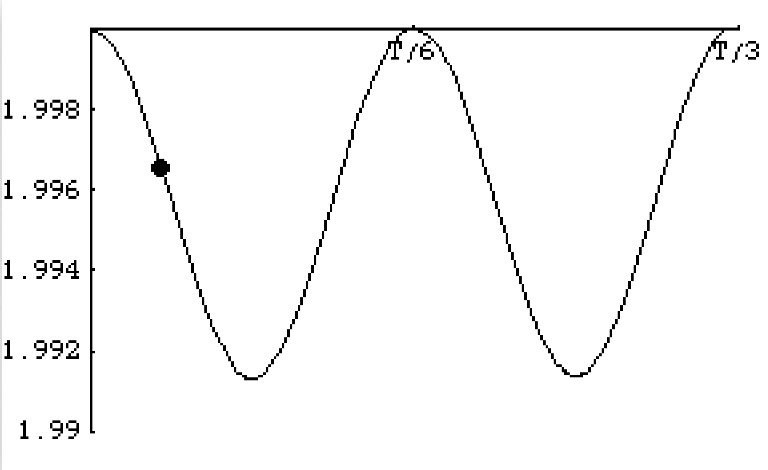
Moment of inertia $I = \sum_i q_i^2$.

Potential energy

$$V_\alpha = \begin{cases} \frac{r^\alpha}{\alpha} & \text{for } \alpha \neq 0 \\ \log r & \text{for } \alpha = 0. \end{cases}$$

Problem (Chenciner). *Show that the moment of inertia I stays constant if and only if $\alpha = -2$.*





$$\frac{\Delta I}{I} \sim \frac{1}{200} \text{ for } \alpha = -1$$

Why 1/r^2 so special? Lagrange-Jacobi identity

$$I = \sum_i q_i^2,\ K = \sum_i \dot{q}_i^2,\ V_\alpha = \frac{1}{\alpha} \sum_{i<j} r_{ij}^\alpha,$$

$$H = \frac{1}{2} K + V_\alpha.$$

$$\Rightarrow \frac{d^2 I}{dt^2} = 2K - 2\alpha V_\alpha = 4E - 2(2+\alpha) V_\alpha$$

For $\alpha = -2$,

$$\frac{d^2 I}{dt^2} = 4E \Rightarrow I = 2Et^2 + c_1 t + c_2.$$

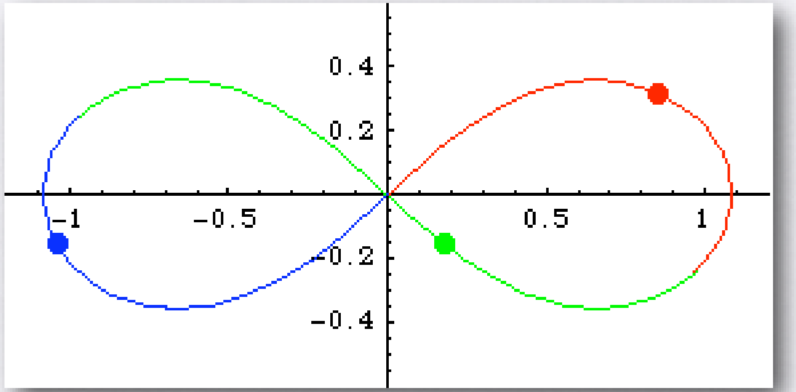
$$\therefore I = \text{const., if } E = 0,\ \frac{dI}{dt}(0) = c_1 = 0.$$

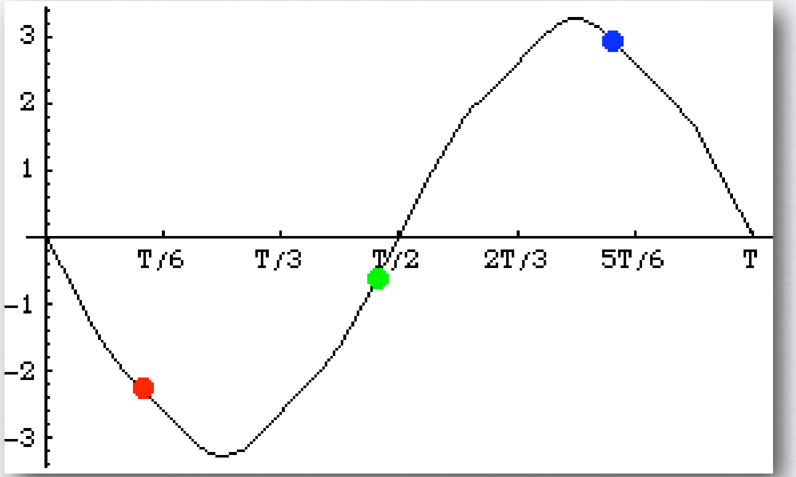
Convexity of Each Lobe (FM)

Theorem (FM). *Each lobe of the eight solution is a convex curve.*

$$\kappa = \frac{\dot{q} \wedge \ddot{q}}{|\dot{q}|^3} = 0 \Leftrightarrow q = 0$$

Computer assisted proof:
T. Kapela & P. Zgliczyński





After my talk
at Math Seminar, Kyoto Univ.,
a man came to me
and said

“３法線も一点で交わりませんか？”

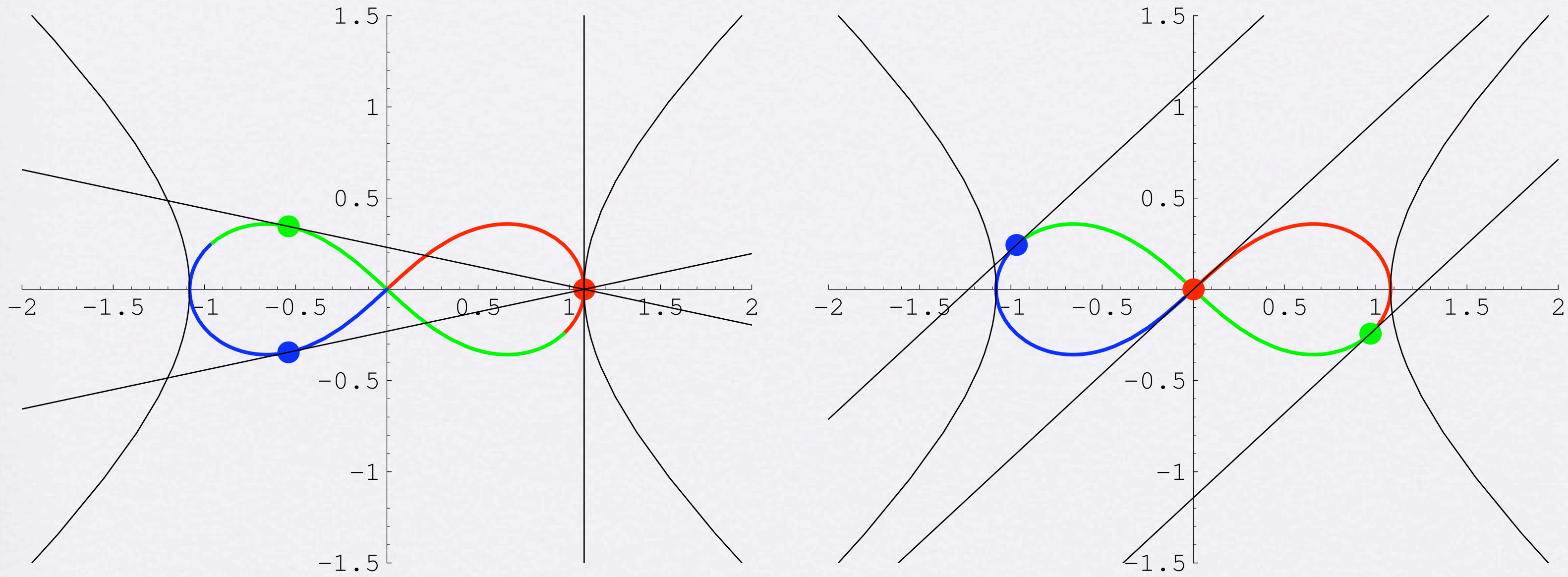
(“Three normal lines meet at a point, don’t they?”)

I said “No, they don’t.”

(Because I know they don’t meet by simulations.)

Then,

Kameyama Pointed out ...



“They meet at a point for Isosceles and Eular configurations.”

On my way home, I have a lot of time to consider...

- Why the three normals do not meet at a point?
- What will happen if they meet at a point?
- What is the differences between tangents and normals?
- etc...

Three Normals Theorem

Theorem (Three Normals). *If* $\sum_i p_i = 0$ *and* $\sum_i q_i \cdot p_i = 0$, *then three normals meet at a point.*

Proof. Let C_n be the crossing point of two normals to q_1 and q_2.

Then, $\sum_i (q_i - C_n) \cdot p_i = 0$,
$(q_1 - C_n) \cdot p_1 = 0$ and
$(q_2 - C_n) \cdot p_2 = 0$.
$\therefore (q_3 - C_n) \cdot p_3 = 0$. $\square$

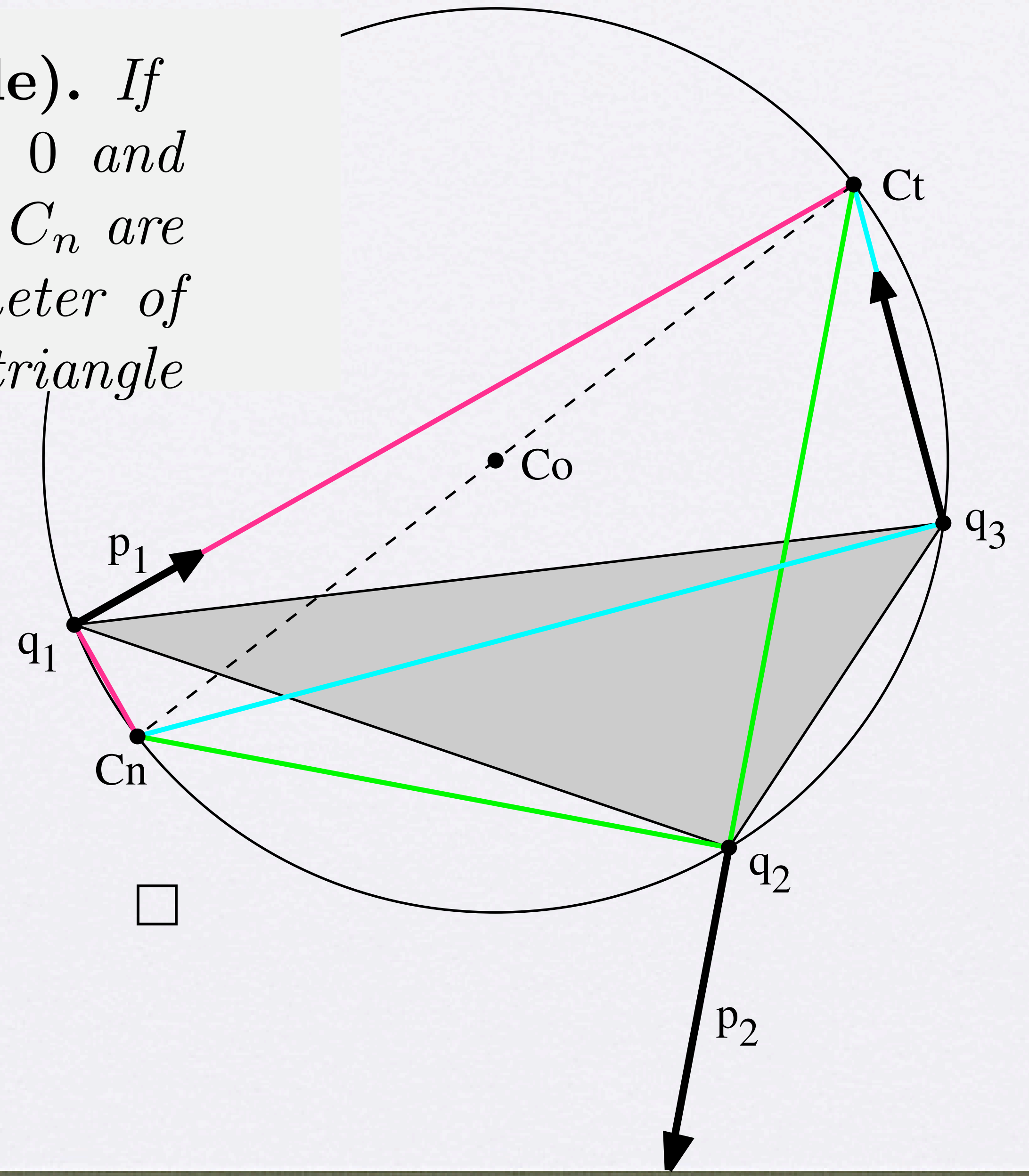
holds for general masses m_i.

C_n: the "Center of Normals"

Then,
Yamada noticed that ...

Circumcircle Theorem

Theorem (CircumCircle). *If* $\sum_i p_i = 0$, $\sum_i q_i \wedge p_i = 0$ *and* $\sum_i q_i \cdot p_i = 0$, *then* C_t *and* C_n *are the end points of a diameter of the circumcircle for the triangle* $q_1 q_2 q_3$.

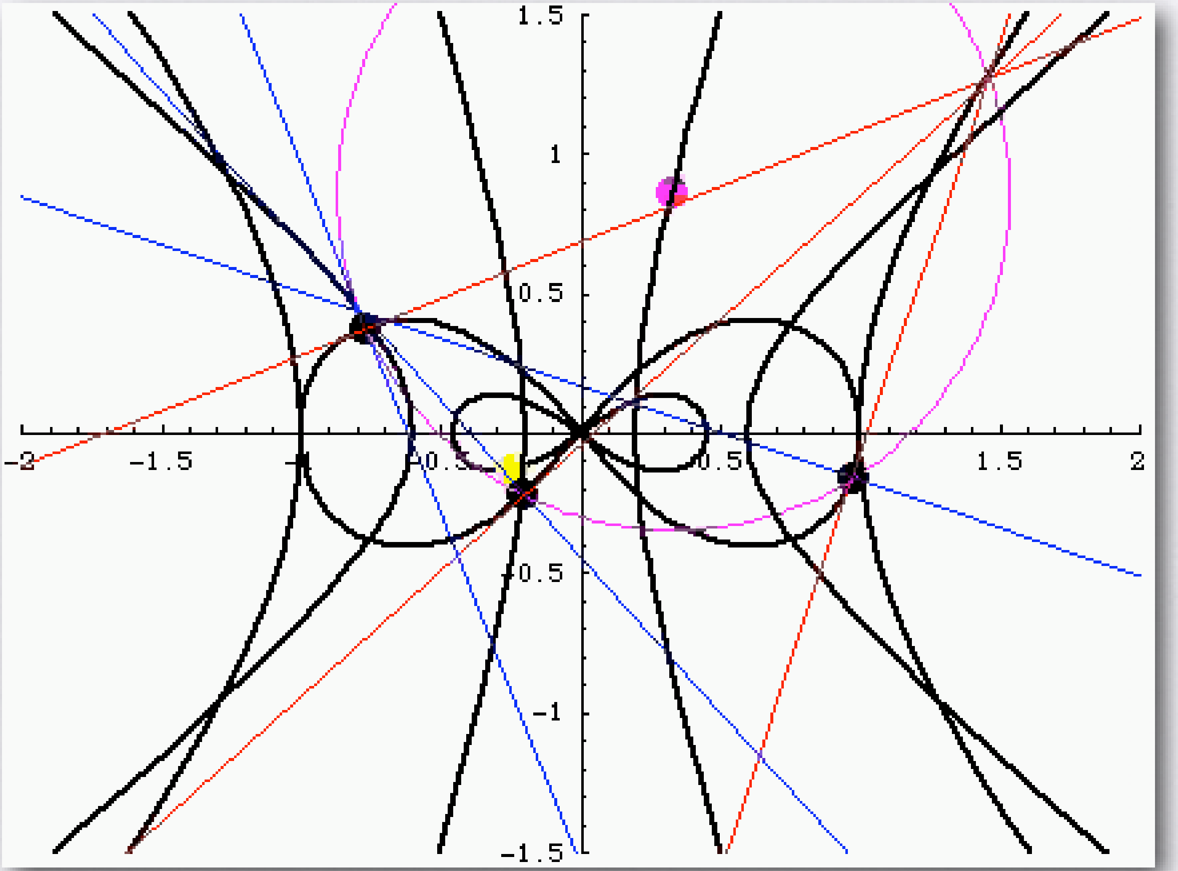


Proof. Angles $C_t q_i C_n$ are 90 degrees for $i = 1, 2, 3$. □

holds for any masses m_i.

Centres for figure-eight solution under 1/r^2 potential

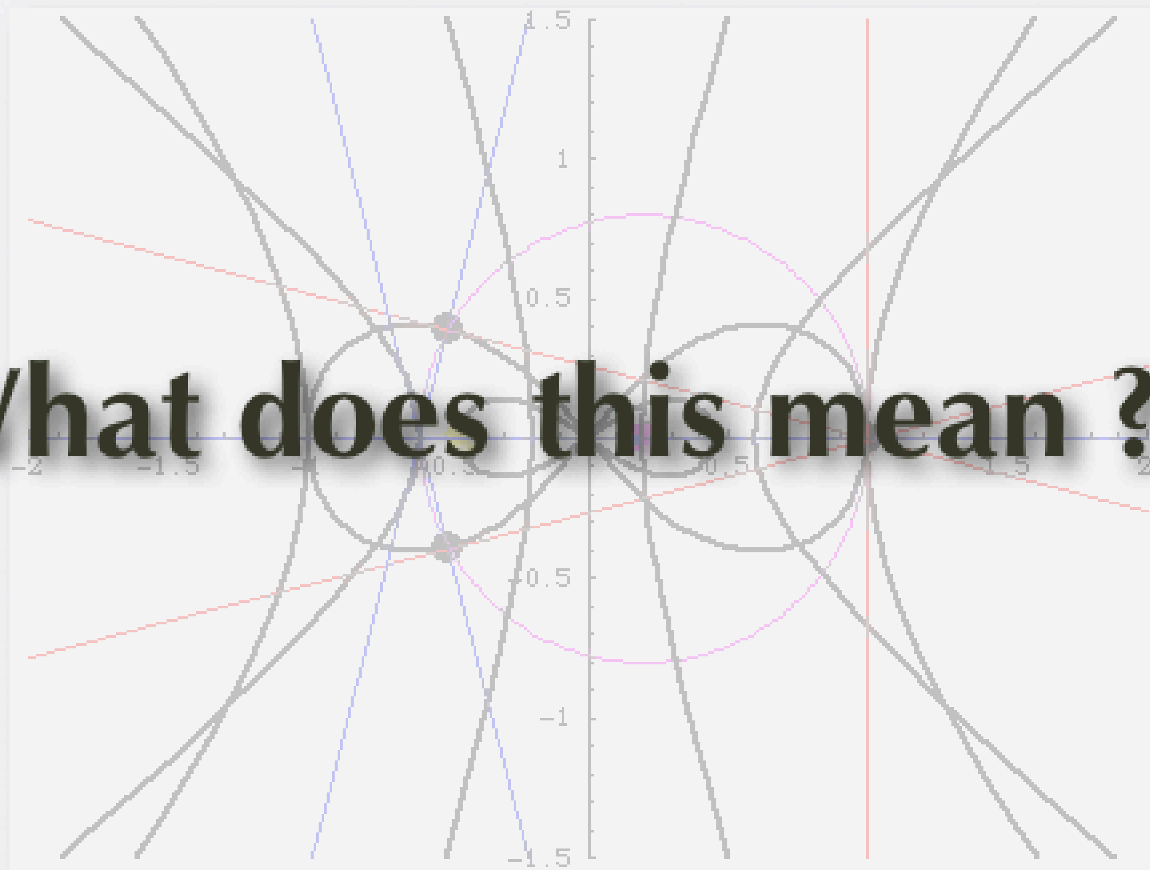
$I = \text{const.}, L = 0$



Purple circle: Circumcircle. Purple point: Circumcenter.
Yellow point: Center of force. Small eight: Orbit of center of force.

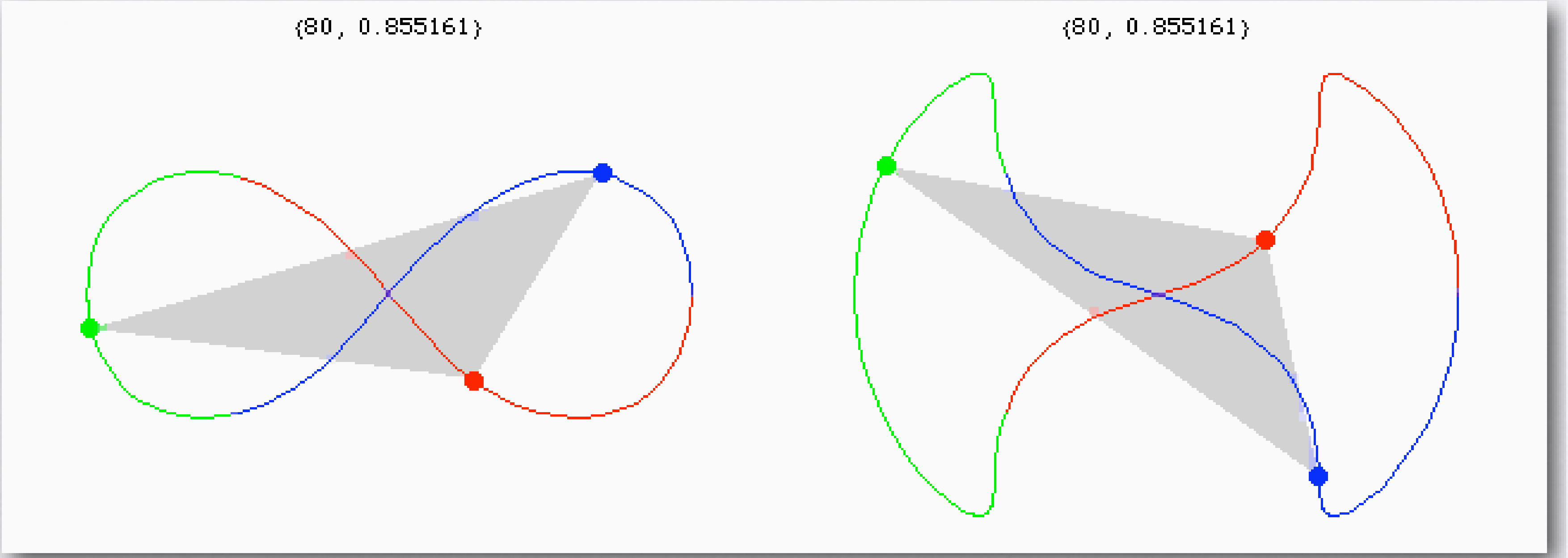
Centres for figure-eight solution under 1/r^2 potential

$I = \text{const.}, L = 0$



What does this mean ?...

Synchronised Triangles for figure-eight under 1/r^2



$$q'_i = \frac{q_i}{\sqrt{I}} \qquad m_i = 1 \qquad p'_i = \frac{p_j - p_k}{\sqrt{3K}}$$

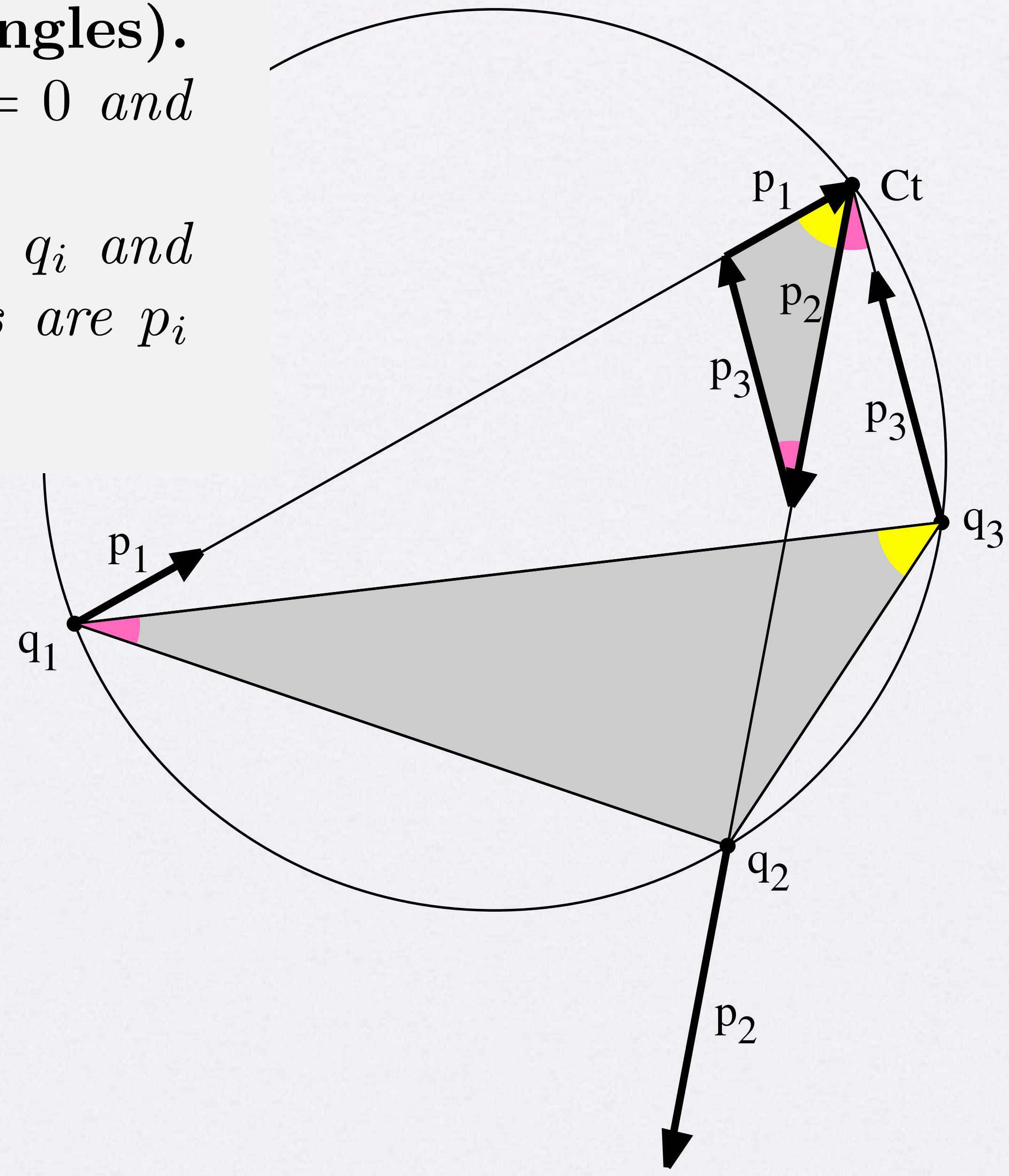
$$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$$

Two triangles are congruent with reverse orientation.

Because ...

Similar Triangles in q & p space

Theorem (Similar Triangles). *If* $\sum_i p_i = 0$, $\sum_i q_i \wedge p_i = 0$ *and* $\sum_i q_i \cdot p_i = 0$, *then triangle whose vetices are* q_i *and triangle whose perimeters are* p_i *are similar with reverse orientation.*



Proof. Look at the angles yellow colored and red colored. It is obvious. □

Remark: *This theorem holds for any masses* m_i

Ratio

$$k(t) = \frac{|p_1|}{|q_2 - q_3|} = \frac{|p_2|}{|q_3 - q_1|} = \frac{|p_3|}{|q_1 - q_2|}$$

$$\therefore \frac{k(t)^2}{m_1 m_2 m_3} = \frac{p_1^2/m_1}{m_2 m_3 (q_2 - q_3)^2} = \frac{p_2^2/m_2}{m_3 m_1 (q_3 - q_1)^2} = \frac{p_3^2/m_3}{m_1 m_2 (q_1 - q_2)^2} = \frac{K}{MI}$$

$$\therefore k(t) = \sqrt{\frac{m_1 m_2 m_3 K}{MI}}$$

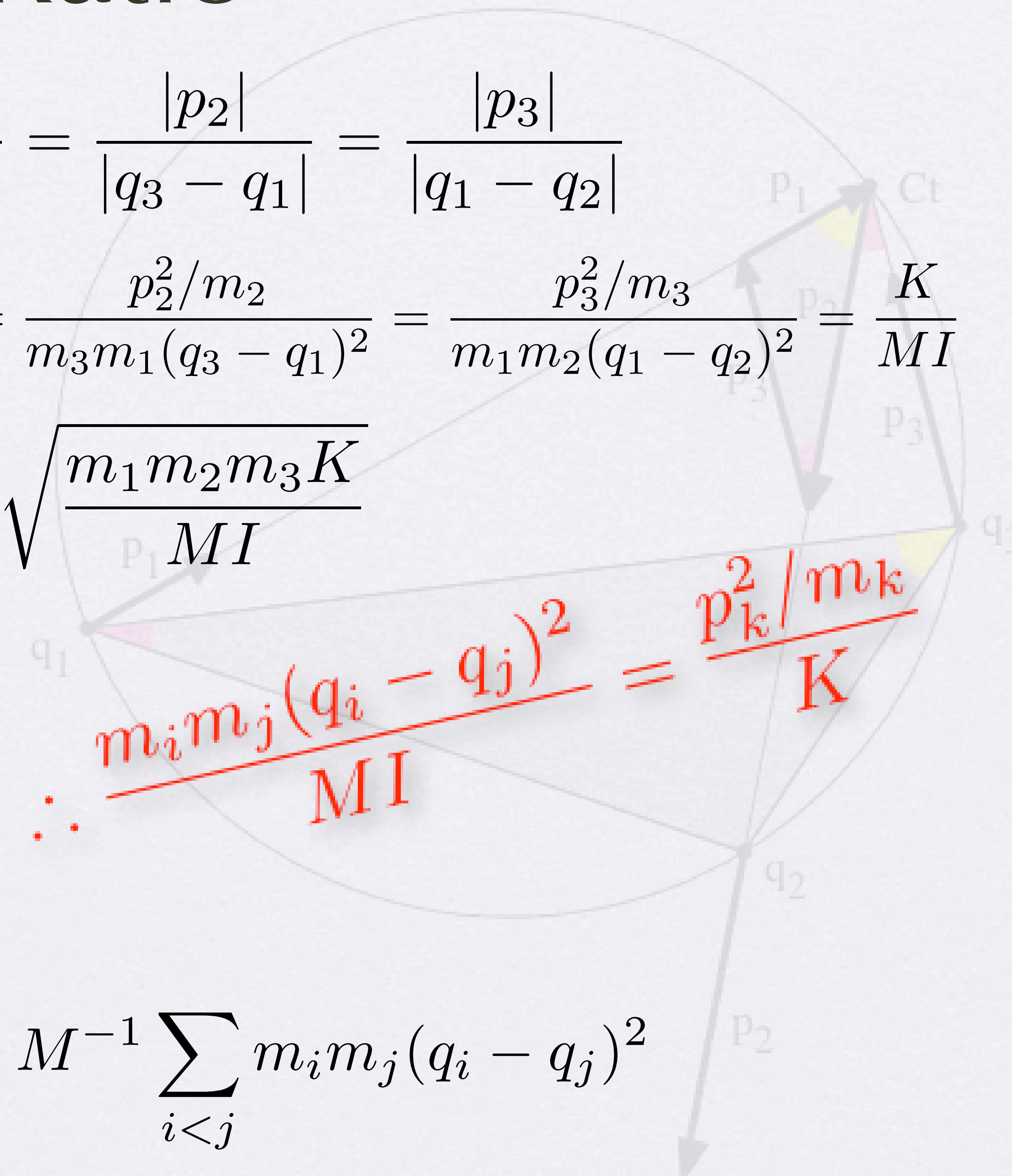
where

$$K = \sum_i p_i^2/m_i$$

$$M = \sum_i m_i$$

$$I = \sum_i m_i q_i^2 = M^{-1} \sum_{i<j} m_i m_j (q_i - q_j)^2$$

$$\therefore \frac{m_i m_j (q_i - q_j)^2}{MI} = \frac{p_k^2/m_k}{K}$$



Similarity in q-v space

Equations we have used are

$$\sum_i m_i q_i = 0, \quad \sum_i m_i v_i = 0,$$

$$\sum_i m_i q_i \cdot v_i = 0, \quad \sum_i m_i q_i \wedge v_i = 0.$$

We have the following three equivalent relations

$$\frac{m_i m_j (q_i - q_j)^2}{MI} = \frac{m_k v_k^2}{K}, \quad \frac{m_k q_k^2}{I} = \frac{m_i m_j (v_i - v_j)^2}{MK},$$

$$\begin{aligned} \frac{m_k q_k^2}{I} + \frac{m_k v_k^2}{K} &= \frac{m_i m_j (q_i - q_j)^2}{MI} + \frac{m_i m_j (v_i - v_j)^2}{MK} \\ &= \frac{m_i + m_j}{M}. \end{aligned}$$

$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2).$

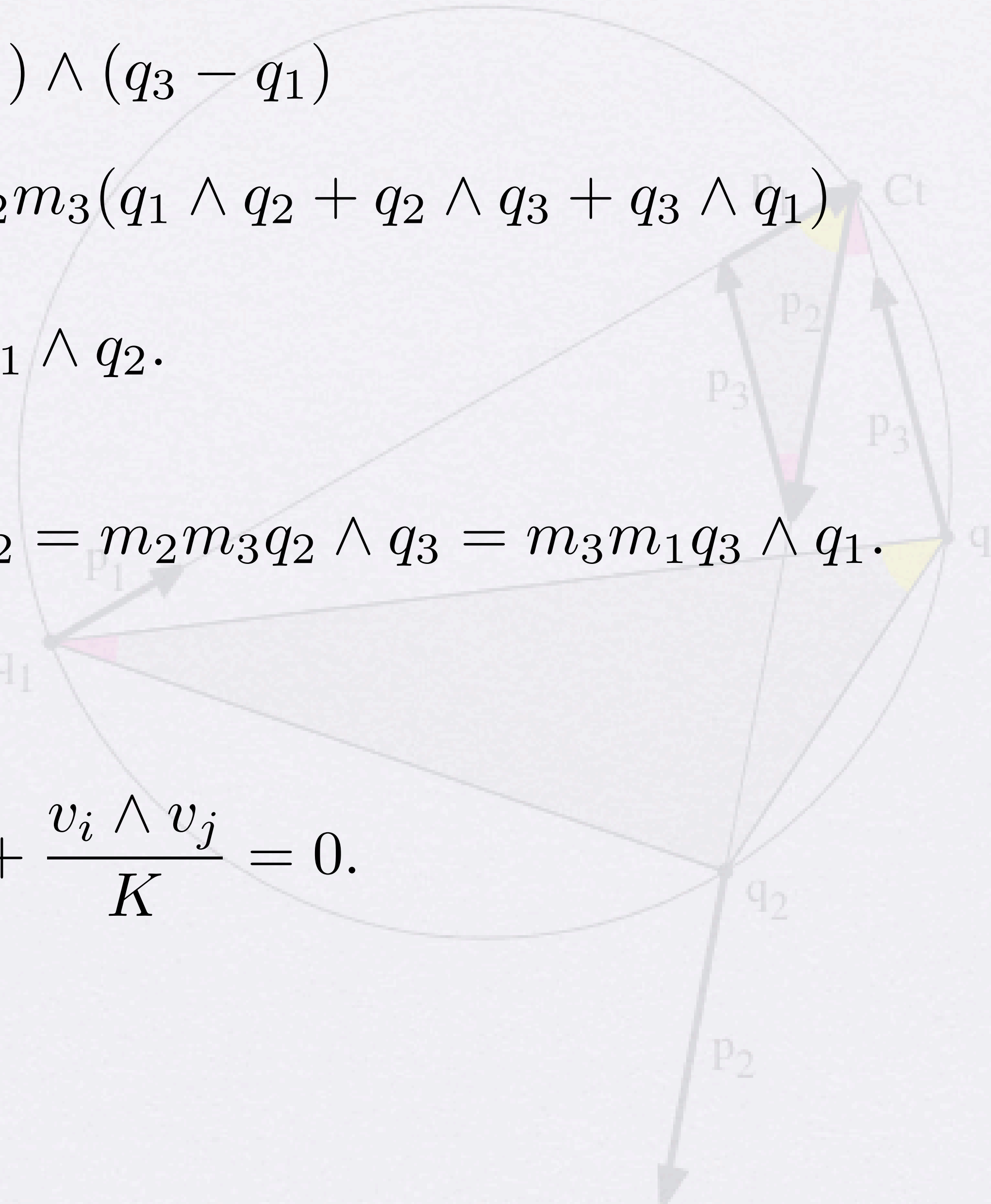
Area

$$
\begin{aligned}
p_1 \wedge p_2 &= -k^2(q_2 - q_1) \wedge (q_3 - q_1) \\
&= -\frac{K}{MI} m_1 m_2 m_3 (q_1 \wedge q_2 + q_2 \wedge q_3 + q_3 \wedge q_1) \\
&= -\frac{K}{I} m_1 m_2 q_1 \wedge q_2.
\end{aligned}
$$

$$\because \sum_i m_i q_i = 0 \Rightarrow m_1 m_2 q_1 \wedge q_2 = m_2 m_3 q_2 \wedge q_3 = m_3 m_1 q_3 \wedge q_1.$$

Therefore, we get

$$\frac{q_i \wedge q_j}{I} + \frac{v_i \wedge v_j}{K} = 0.$$



Energy balance for the orbits under homogeneous potentials

$$\frac{d^2 I}{dt^2} = 0 \Rightarrow \sum_k \frac{p_k^2}{m_k} = \sum_{i<j} m_i m_j r_{ij}^{\alpha}$$

So far, we do not use the explicit form of the potential.
We assumed only the existence the orbits
with L=0 and dI/dt=0.

What will happen for orbits under 1/r^2 potential?

What will happen if L=0 and dI/dt=0 orbits are allowed
under the other homogeneous potentials?

Energy balance for the orbits under 1/r^2

$$\frac{d^2 I}{dt^2} = 0 \Rightarrow K = \sum_{i<j} \frac{m_i m_j}{r_{ij}^2}$$

$$L = 0, \quad \frac{dI}{dt} = 0 \Rightarrow \frac{1}{r_{ij}^2} = \frac{m_1 m_2 m_3 K}{MI} \, \frac{1}{p_k^2}.$$

$$\therefore K = \frac{m_1 m_2 m_3 K}{MI} \left(\frac{m_1 m_2}{p_3^2} + \frac{m_2 m_3}{p_1^2} + \frac{m_3 m_1}{p_2^2} \right)$$

$$\therefore \frac{m_1 m_2}{p_3^2} + \frac{m_2 m_3}{p_1^2} + \frac{m_3 m_1}{p_2^2} = \frac{MI}{m_1 m_2 m_3} = \text{const.}$$

Energy balance for the orbits under log r

$$\frac{d^2 I}{dt^2} = 0 \Rightarrow \qquad K = \sum_k \frac{p_k^2}{m_k} = \sum_{i<j} m_i m_j,$$

$$V_0 = \sum_{i<j} m_i m_j \log r_{ij} = E - \frac{1}{2} \sum_{i<j} m_i m_j.$$

$$\therefore L = 0, \quad \frac{dI}{dt} = 0 \Rightarrow \sum_{ijk} m_i m_j \log |p_k| = E + \frac{1}{2} \log \frac{m_1 m_2 m_3 K}{MI}.$$

Energy balance for the orbits under other homogeneous potentials

For $\alpha \neq 0, -2$,

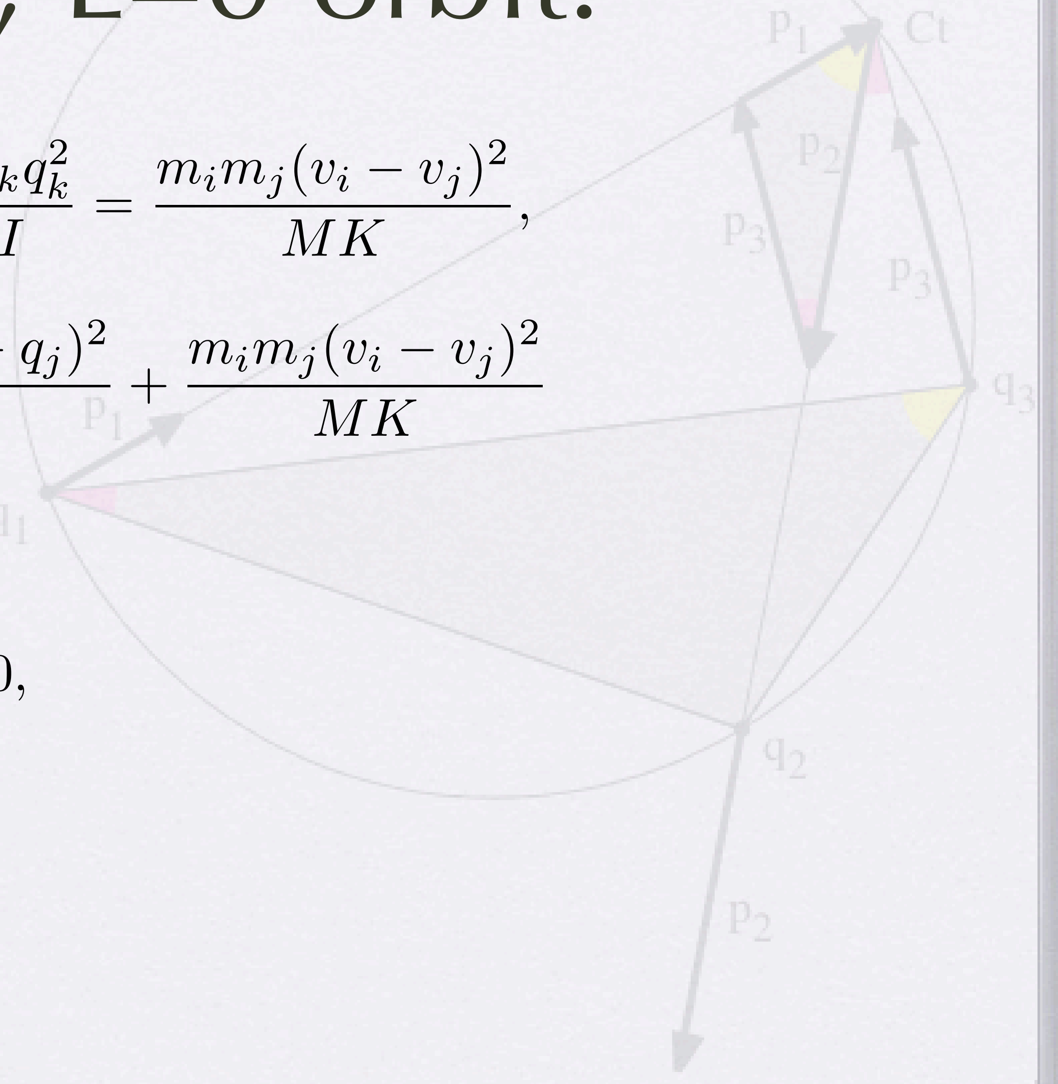
$$\frac{d^2 I}{dt^2} = 0 \Rightarrow K = \sum_k \frac{p_k^2}{m_k} = \sum_{i<j} m_i m_j r_{ij}^\alpha = \frac{\alpha E}{2+\alpha}$$

$$L = 0, \quad \frac{dI}{dt} = 0 \Rightarrow \sum_{ijk} m_i m_j |p_k|^\alpha = K \left(\frac{m_1 m_2 m_3 K}{MI} \right)^{\frac{\alpha}{2}} = \text{const.}$$

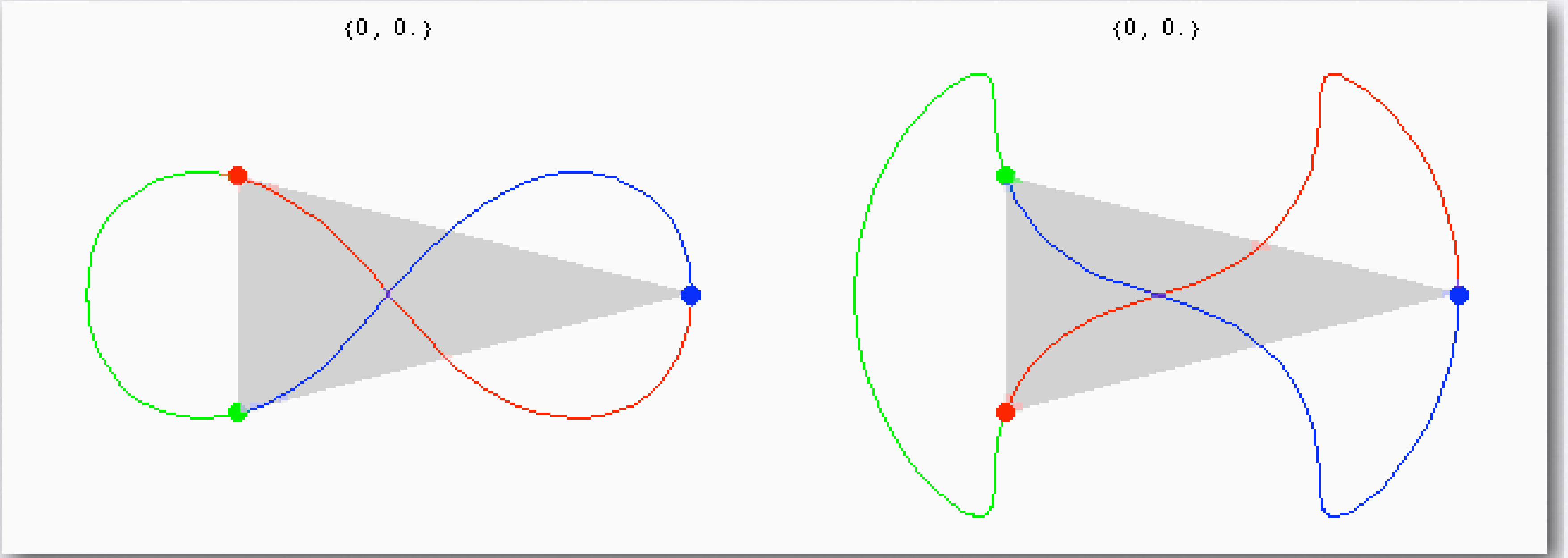
$\Rightarrow$ *Conceptional proof of the "Chenciner's Problem"?*
"No figure-eight with $I = \text{const.}$ exept for $\alpha = -2$".

Conclusion 1: Synchronised Triangles for I=const., L=0 orbit.

$$
\begin{aligned}
\frac{m_i m_j (q_i - q_j)^2}{MI} &= \frac{m_k v_k^2}{K}, \quad \frac{m_k q_k^2}{I} = \frac{m_i m_j (v_i - v_j)^2}{MK}, \\
\frac{m_k q_k^2}{I} + \frac{m_k v_k^2}{K} &= \frac{m_i m_j (q_i - q_j)^2}{MI} + \frac{m_i m_j (v_i - v_j)^2}{MK} \\
&= \frac{m_i + m_j}{M}, \\
\frac{q_i \wedge q_j}{I} &+ \frac{v_i \wedge v_j}{K} = 0, \\
\sum_{ijk} m_i m_j |p_k|^\alpha &= \text{const.}
\end{aligned}
$$



Conclusion 2: Synchronised Triangles for figure-eight under 1/r^2



$$q_i' = \frac{q_i}{\sqrt{I}} \qquad m_i = 1 \qquad p_i' = \frac{p_j - p_k}{\sqrt{3K}}$$

Two triangles are congruent with reverse orientation.

$$\sum_i \frac{1}{p_i^2} = 3I$$

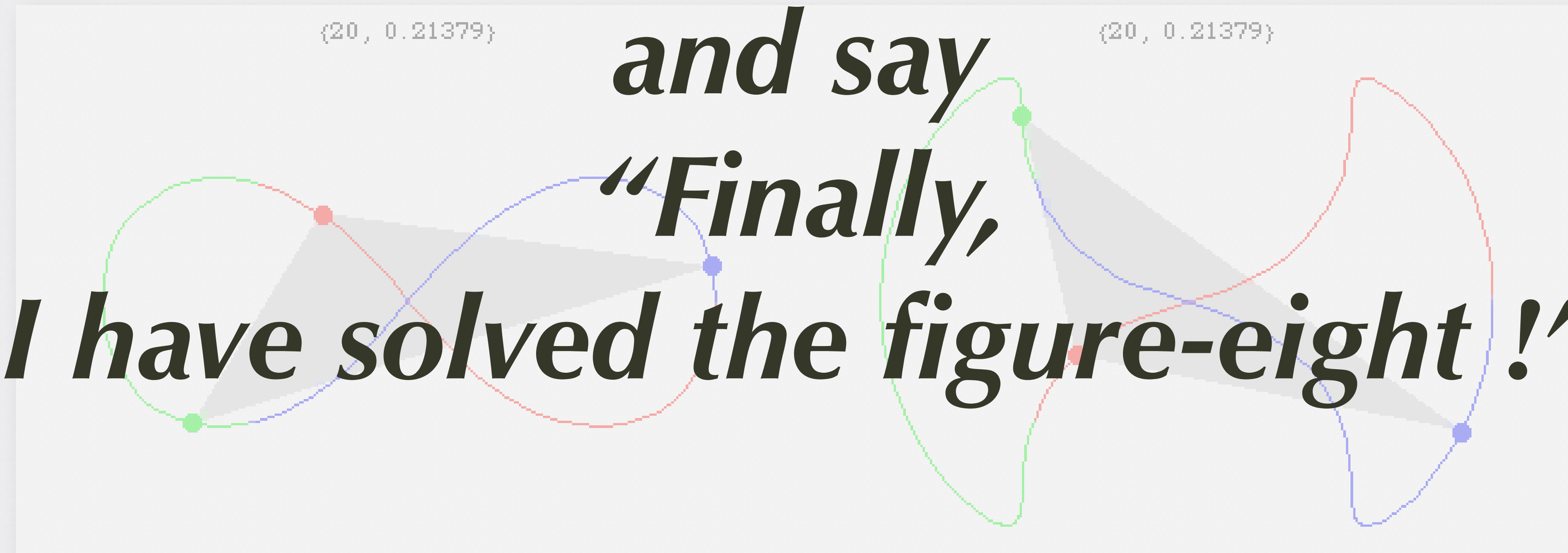
I have a dream.

One day, someone mail me

and say

"Finally,

I have solved the figure-eight !"



Thank you.