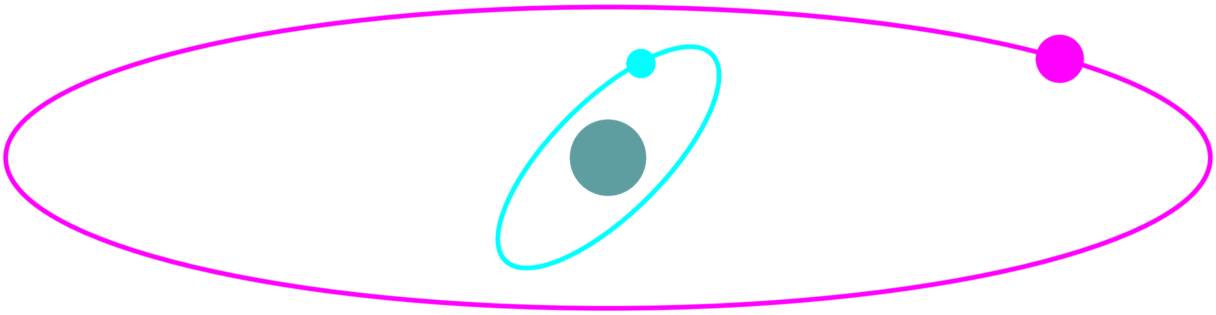
Kozai Mechanism (“Resonance”)

An Introduction to the Secular Perturbation Theory



Eiichiro Kokubo

National Astronomical Observatory

[References]

Kozai (1962)

Secular Perturbation of Asteroids
with High Inclination and Eccentricity
AJ, 67, 591

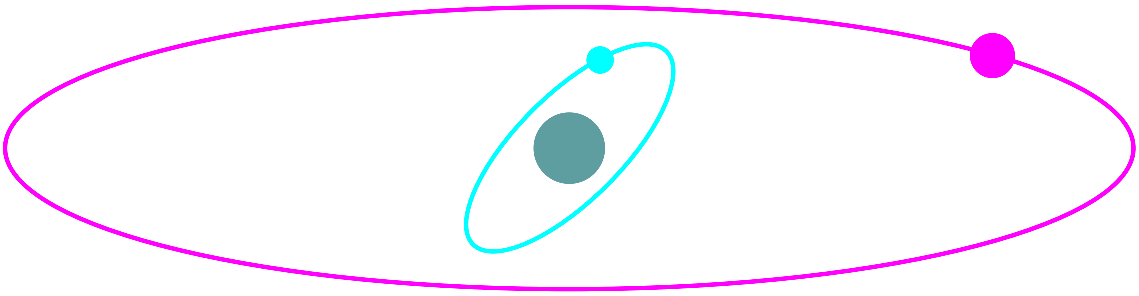
Kinoshita & Nakai (1991)

Secular Perturbation of Fictitious Satellites of Uranus
Celest. Mech. & Dyn. Astr., 52, 293

Kinoshita & Nakai (1999)

Analytical Solution of the Kozai Resonance
and its Application
Celest. Mech. & Dyn. Astr., 75, 125

[Model]



Restricted Three-Body Problem

primary

- mass: m
- position: coordinate origin

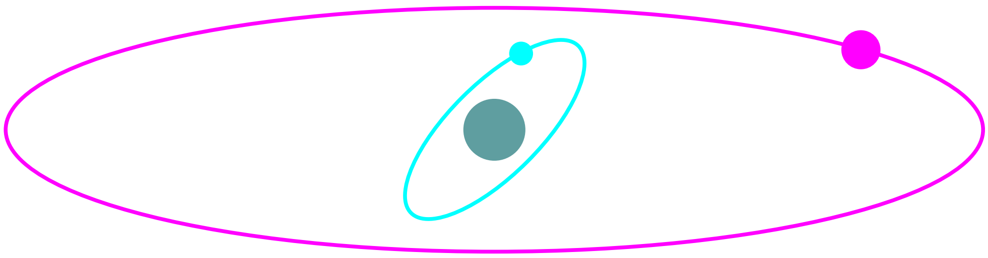
secondary

- mass: m'
- position: $\boldsymbol{r}'$
- orbital elements: $a', e', i' = 0, \Omega', \omega', \varpi', n', l', f'$

test particle

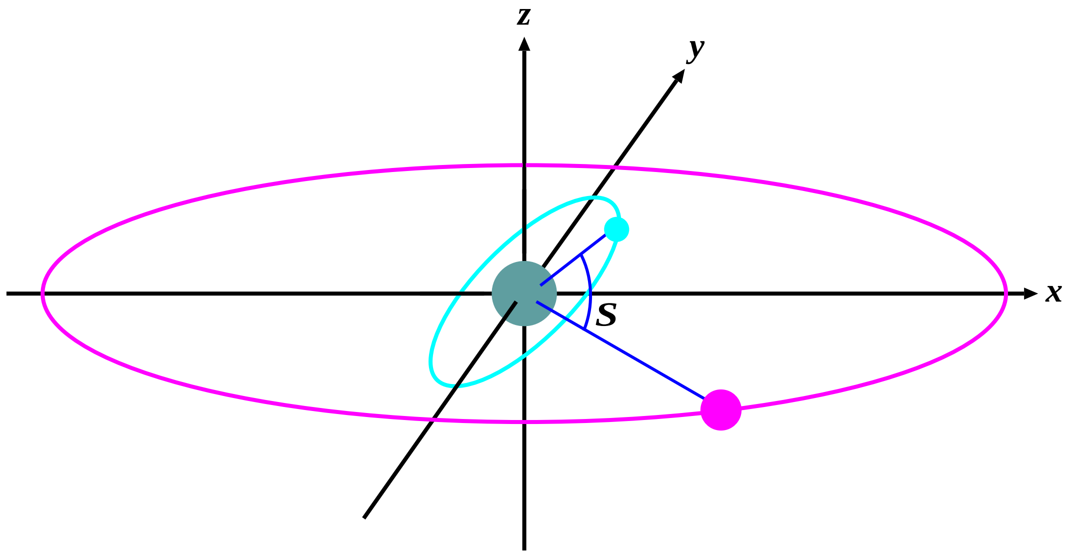
- mass: 0 (mass-less)
- position: $\boldsymbol{r}$ ($r < r'$)
- orbital elements: $a, e, i, \Omega, \omega, \varpi, n, l, f$

[Application]



test particle	primary	secondary
solar system		
asteroid	sun	planet
comet	sun	planet
planet	sun	planet
satellite system		
satellite	planet	sun
extrasolar planetary system		
planet	star	star
planet	star	planet

[Equation of Motion]



Equation of Motion

$$\frac{d^2\boldsymbol{r}}{dt^2} = -k^2 m \frac{\boldsymbol{r}}{r^3} + \frac{\partial R}{\partial \boldsymbol{r}}$$

Disturbing Function

$$R = k^2 m' \left(\underbrace{\frac{1}{\Delta}}_{\text{direct term}} - \underbrace{\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r'^3}}_{\text{indirect term}} \right)$$

$$\Delta^2 = (x - x')^2 + (y - y')^2 + z^2 = r^2 + r'^2 - 2rr' \cos S$$

$$\cos S = \cos(\omega + f)\cos(\omega' + f') + \sin(\omega + f)\sin(\omega' + f')\cos i$$

Hamiltonian

$$F = -\frac{k^2 m}{2a} - R$$

[Expansion of Disturbing Function]

Assumption

$$r' \gg r$$

Expansion with Legendre Polynomials

$$\begin{aligned}\frac{r'}{\Delta} &= \left[1 + \left(\frac{r}{r'}\right)^2 - 2\frac{r}{r'}\cos S\right]^{-1/2} \\ &= 1 + \left(\frac{r}{r'}\right) P_1(\cos S) + \left(\frac{r}{r'}\right)^2 P_2(\cos S) + \ldots\end{aligned}$$

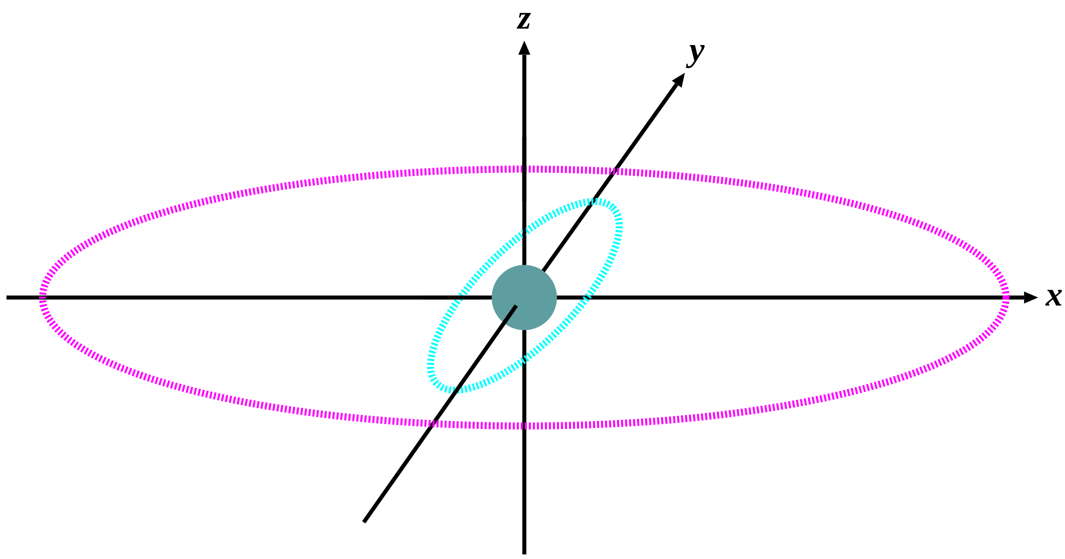
Expanded R up to $(r/r')^2$

$$\begin{aligned}R &\simeq \frac{k^2 m'}{r'}\left[\left(\frac{r}{r'}\right)^2 P_2(\cos S)\right] \\ &= \frac{k^2 m' a^2}{a'^3}\left[\left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \left(-\frac{1}{2} + \frac{3}{2}\cos^2 S\right)\right]\end{aligned}$$

0th: no contribution

1st: cancel with the indirect term

[Orbit Average of Disturbing Function]



Orbit Average

$$\frac{1}{T}\int_0^T dt = \frac{1}{2\pi}\int_0^{2\pi} dl \equiv \langle\rangle_l$$

Orbit-Averaged R

$$\langle\langle R\rangle_{l'}\rangle_l \simeq \frac{k^2 m' a^2}{a'^3}\left[-\frac{1}{2}\left\langle\left\langle\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3\right\rangle_{l'}\right\rangle_l + \frac{3}{2}\left\langle\left\langle\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3 \cos^2 S\right\rangle_{l'}\right\rangle_l\right]$$

$$= \frac{m'}{m+m'}\frac{n'^2 a^2}{\eta'^3}\left[\frac{1}{8}\left(1+\frac{3}{2}e^2\right)\left(3\cos^2 i - 1\right) + \frac{15}{16}e^2 \cos 2\omega \sin^2 i\right]$$

Axisymmetric ring potential (no Ω and Ω' in $\langle R\rangle$)

Angular Momentum Conservation

$$L_z = \sqrt{k^2 ma(1-e^2)}\cos i = \text{const.}$$

[Lagranges's Planetary Equations]

Lagranges's Planetary Equations

$$
\begin{aligned}
\frac{da}{dt} &= 0 \\
\frac{de}{dt} &= -\frac{\eta}{na^2 e}\frac{\partial \langle R \rangle}{\partial \omega} \\
\frac{di}{dt} &= \frac{\cot i}{na^2 \eta}\frac{\partial \langle R \rangle}{\partial \omega} \\
\frac{d\omega}{dt} &= \frac{\eta}{na^2 e}\frac{\partial \langle R \rangle}{\partial e} - \frac{\cot i}{na^2 \eta}\frac{\partial \langle R \rangle}{\partial i} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2 \eta \sin i}\frac{\partial \langle R \rangle}{\partial i}
\end{aligned}
$$

$\dot{a} = 0 \Longrightarrow$ no secular change in a

Condition for "Kozai resonance"

$$\dot{\omega} = 0 \Longrightarrow \dot{\varpi} = \dot{\Omega}$$

[Hamiltonian Map]

Orbit-Averaged Hamiltonian

$$\langle F \rangle = -\frac{k^2 m}{2\langle a \rangle} - \langle R \rangle = \text{const.} \Longrightarrow \langle R \rangle = \text{const.}$$

"Angular Momentum" Conservation

$$(1 - e^2)\cos^2 i = \text{const.} \equiv h$$

$$0 < i < \pi/2 \quad e_{\text{max}} \leftrightarrow i_{\text{min}}$$

$$\pi/2 < i < \pi \quad e_{\text{max}} \leftrightarrow i_{\text{max}}$$

"Energy" Conservation

$$\begin{aligned} R^* &= (2 + 3e^2)(3\cos^2 i - 1) + 15e^2 \cos 2\omega \sin^2 i \\ &= (2 + 3e^2)\left(\frac{3h}{1 - e^2} - 1\right) + 15e^2\left(1 - \frac{h}{1 - e^2}\right)\cos 2\omega \\ &\equiv c \end{aligned}$$

$$c, h \implies R^*(e, \omega) = c$$

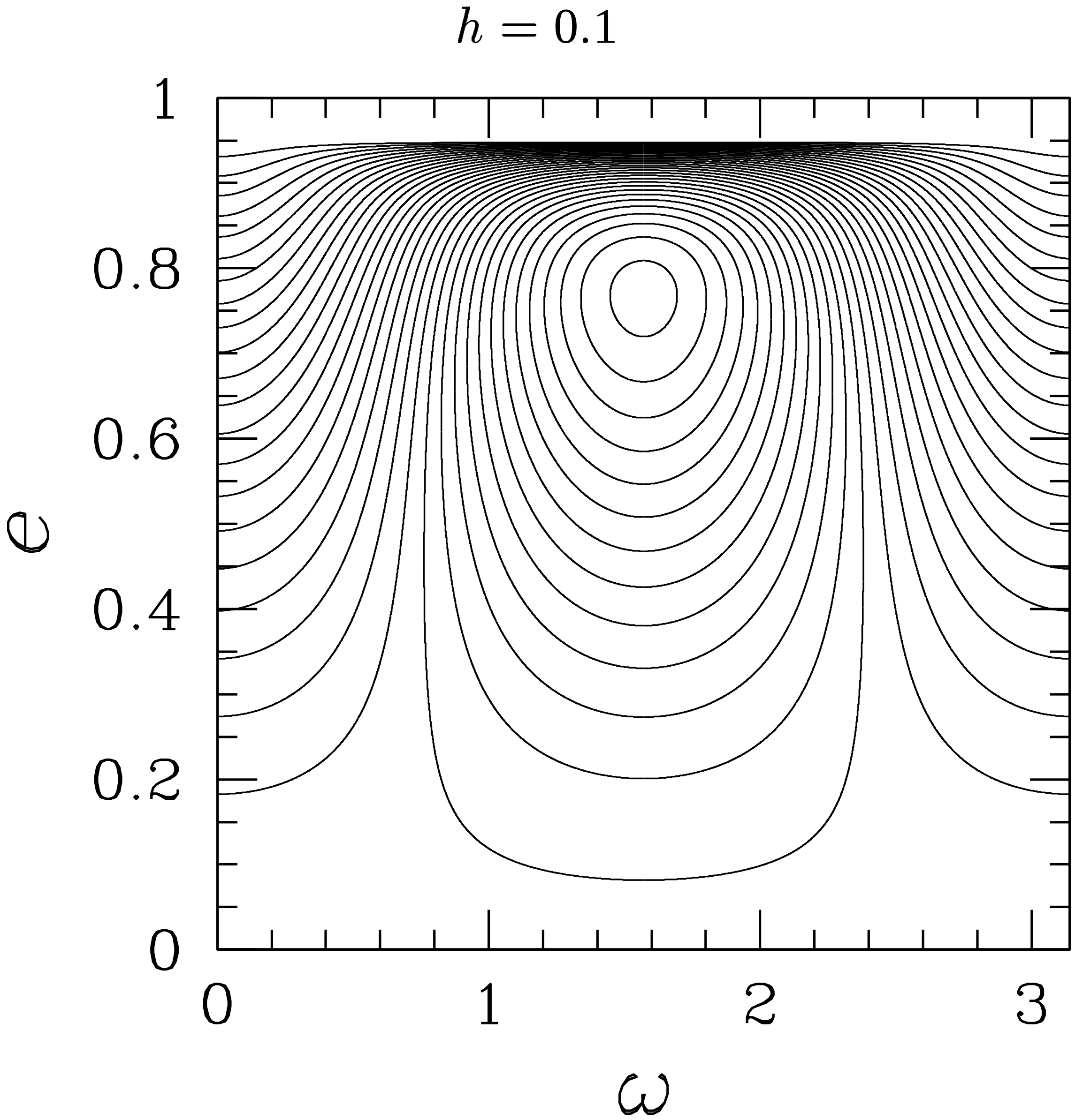
Contour

$$\omega = \frac{1}{2}\arccos\alpha$$

$$\alpha = \frac{c(1 - e^2) - (2 + 3e^2)(3h - 1 + e^2)}{15e^2(1 - e^2 - h)}$$

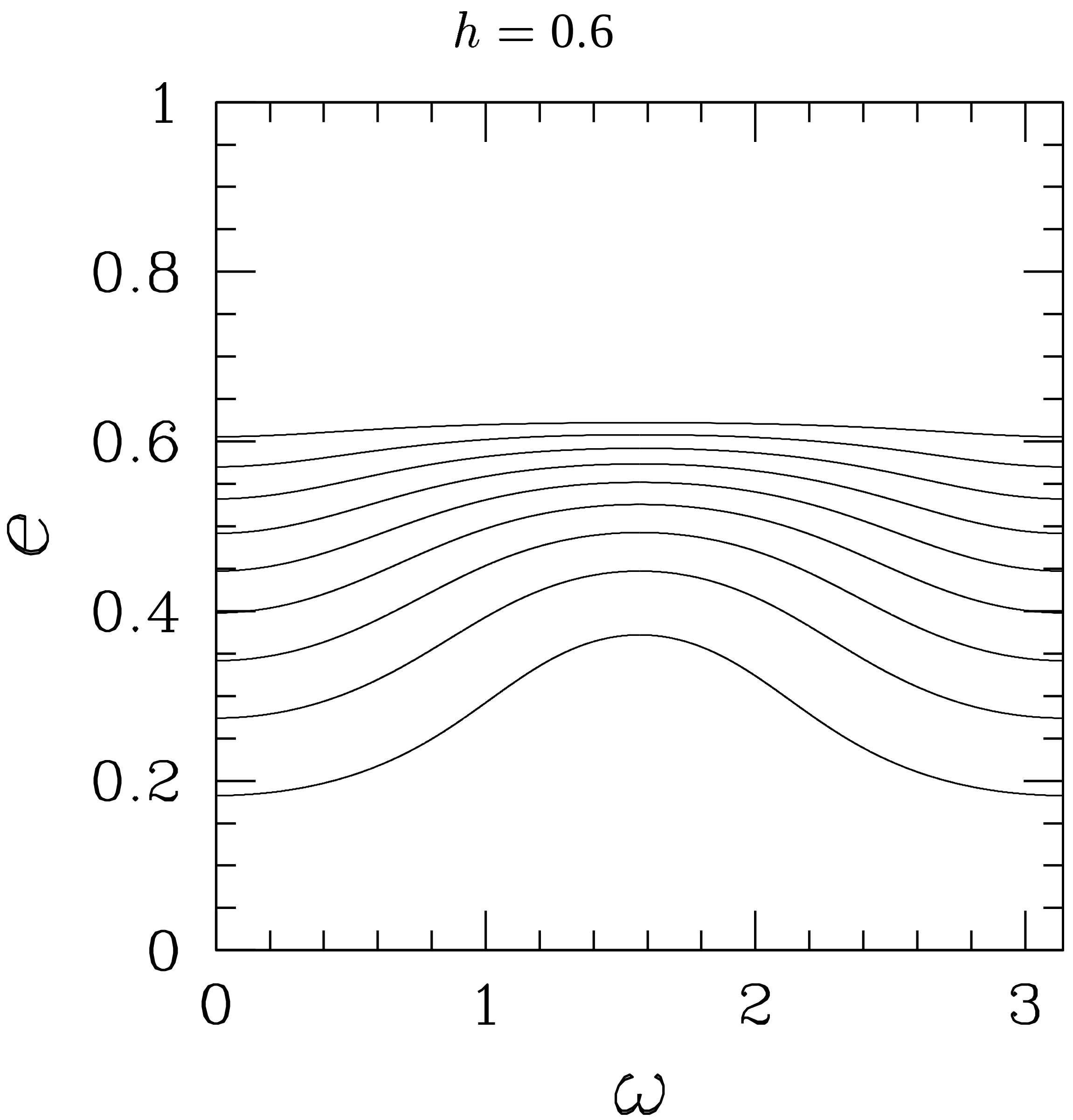
$|\alpha| < 1 \implies$ possible e

[Hamiltonian Map –Libration–]



- Libration around $\omega = \pi/2 (3\pi/2)$
- e becomes large for orbits near separatrix

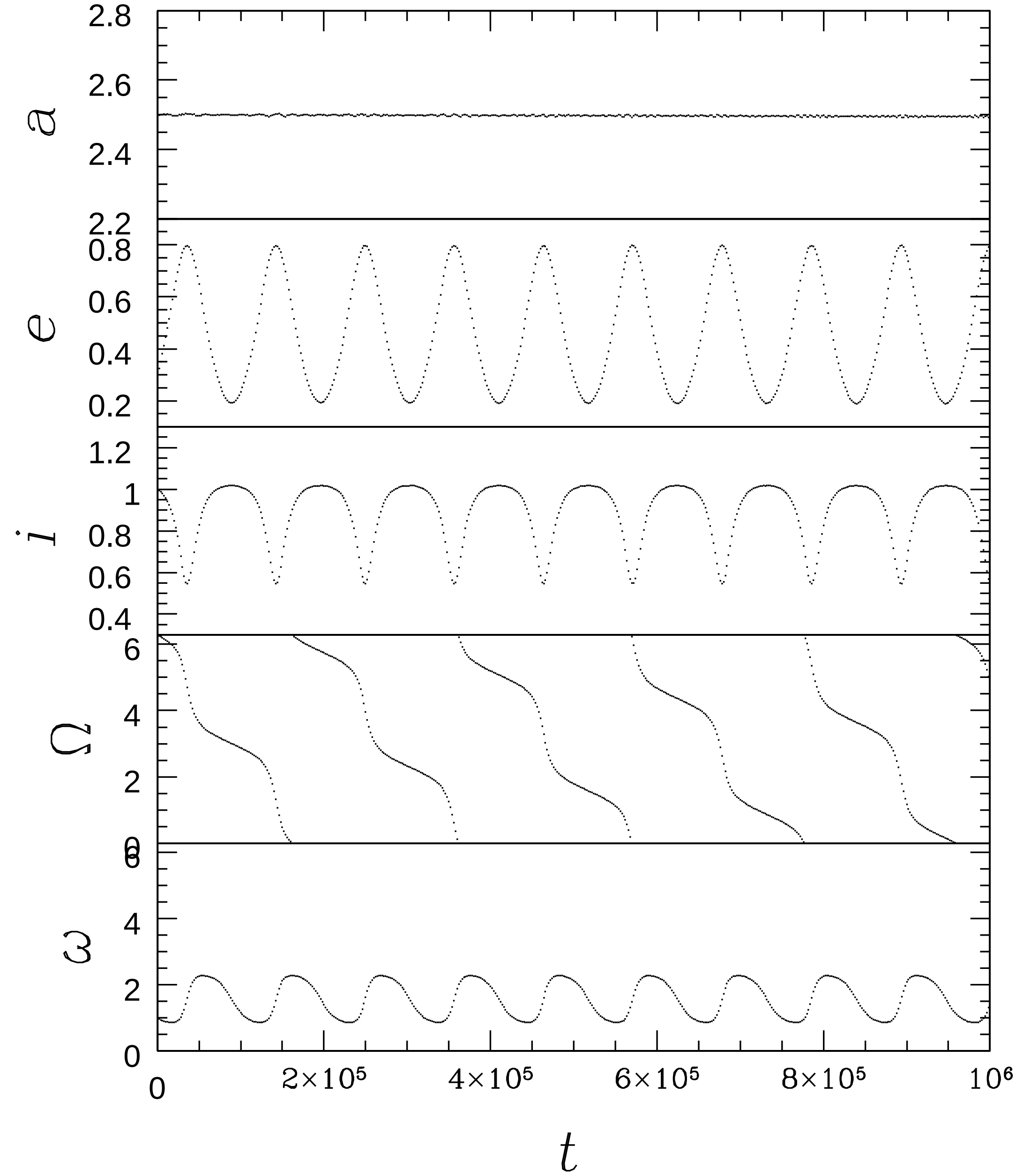
[Hamiltonian Map –Circulation–]



[Numerical Integration –Libration–]

$$m' = 0.001m, a' = 5, e' = 0$$

$$a = 2.5, e = 0.3, i = 1.0, \omega = 1.0 \Longrightarrow h = 0.27, c = -0.68$$



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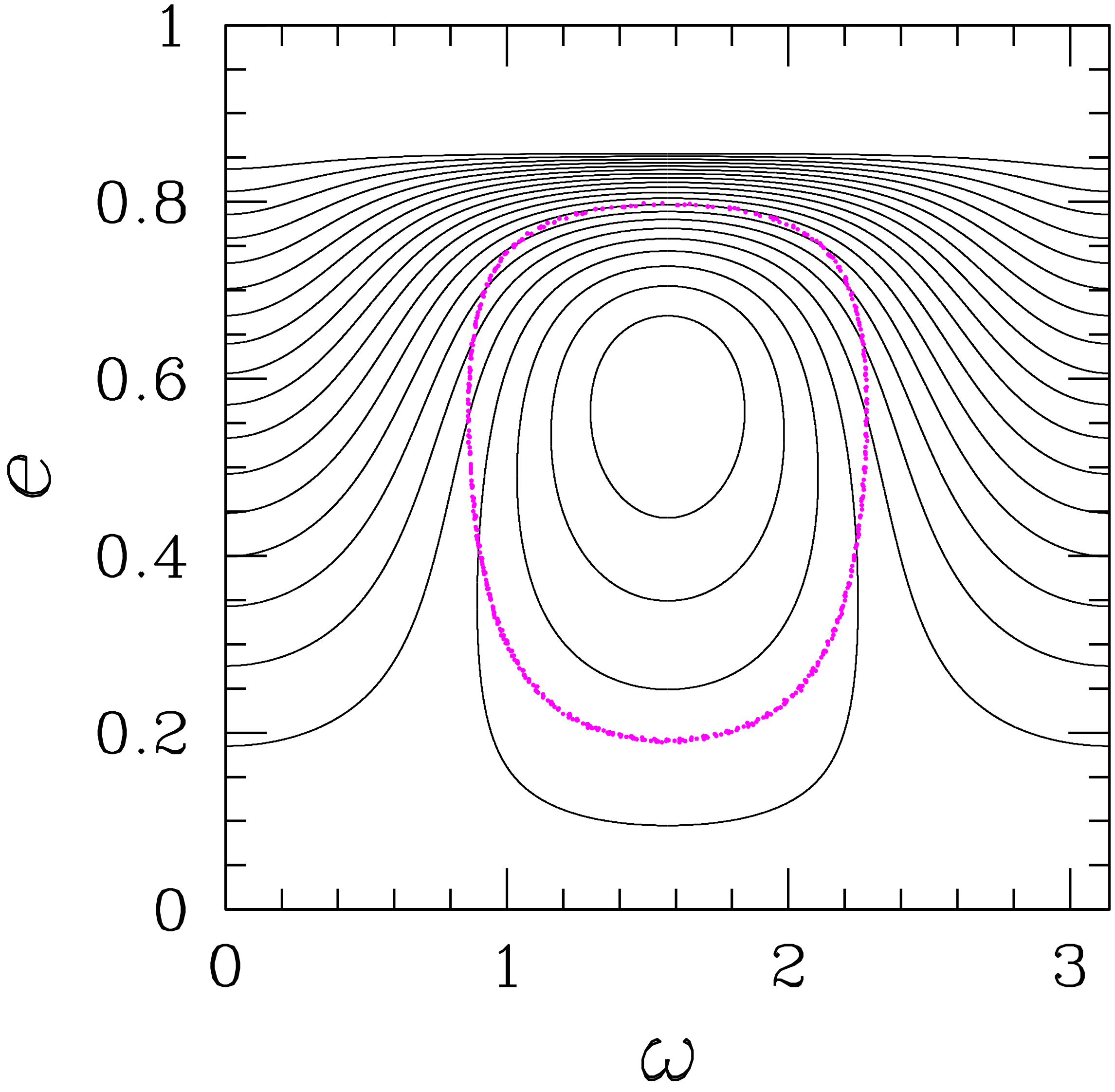

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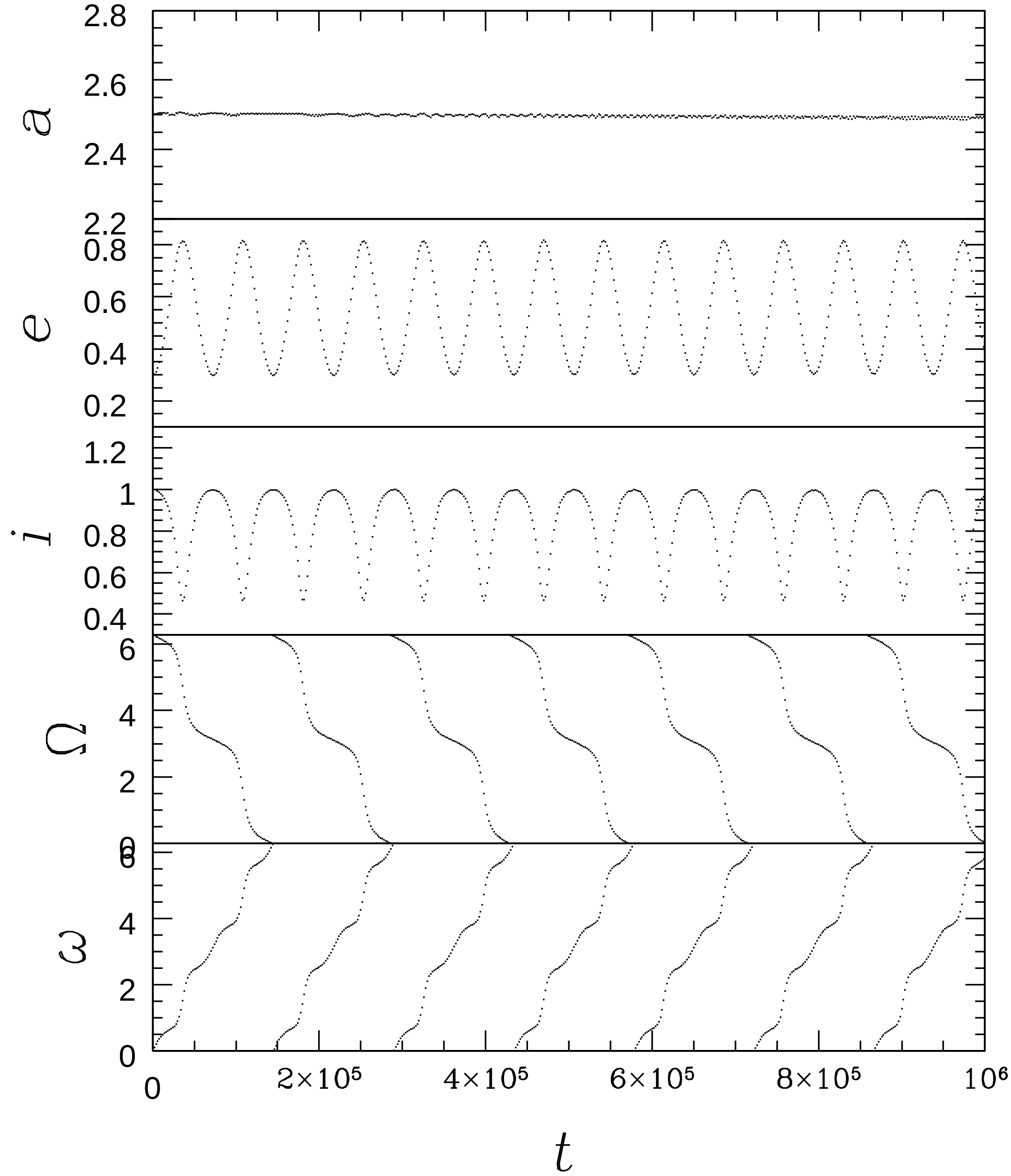

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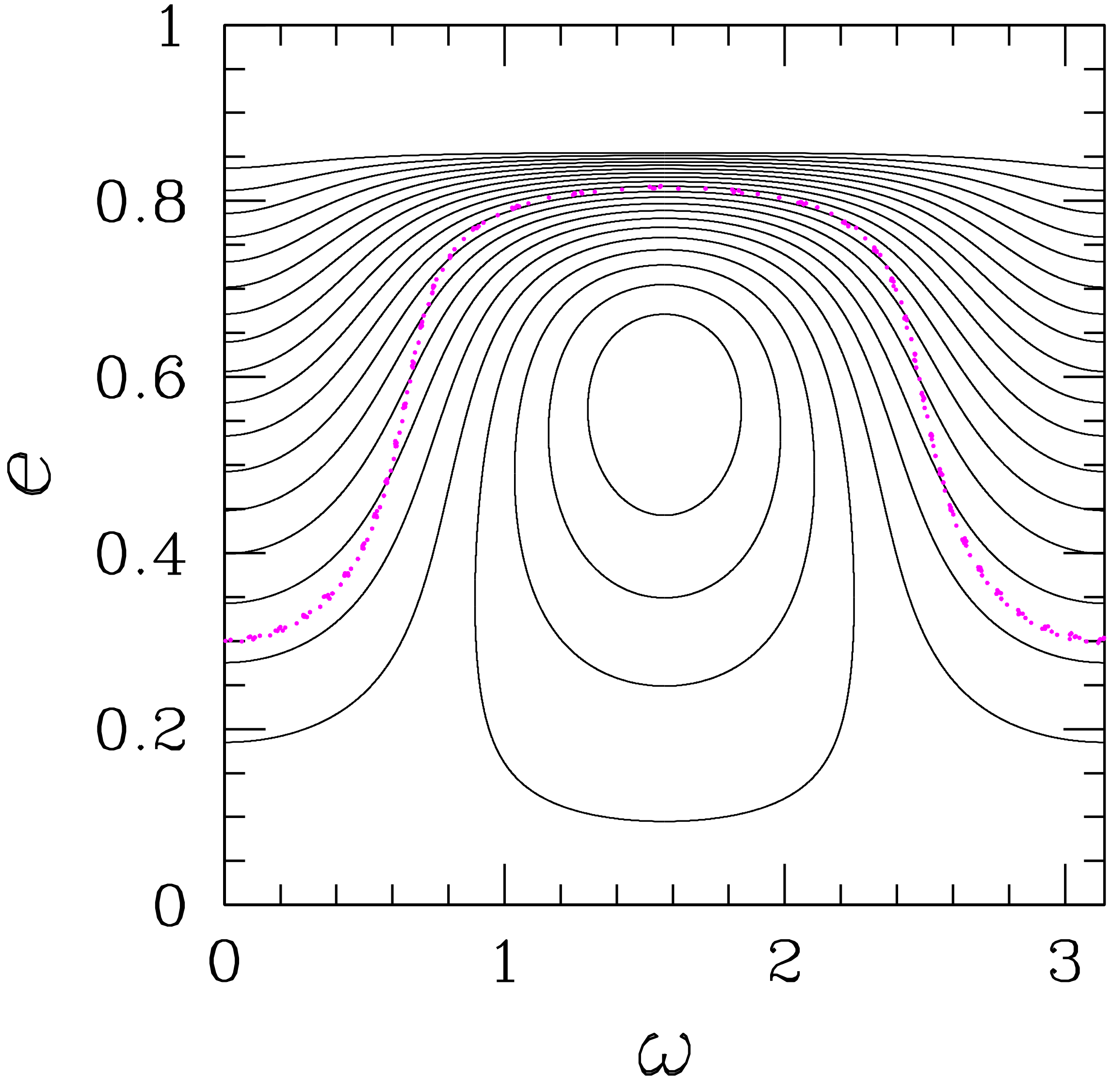
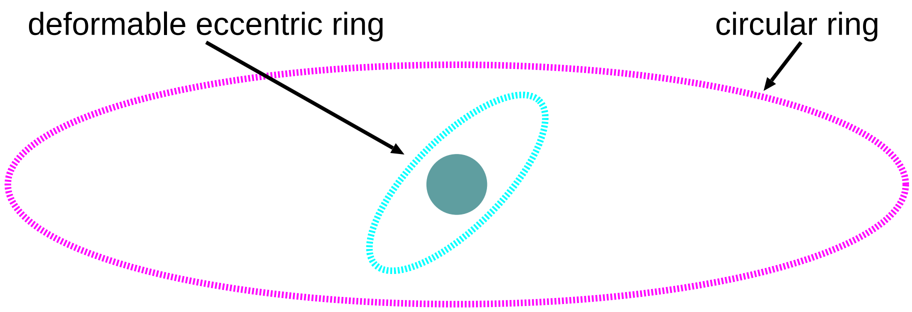


[Summary]



2nd-Order Secular Perturbation

- Axisymmetric perturbation
  (angular momentum conservation)
- No secular change in energy
  (energy conservation)

Kozai Mechanism (Resonance)

- ω-libration around $\pi/2 (3\pi/2)$
- Not a secular resonance

Other Cases

- Kozai mechanism with an inner perturber
- Kozai mechanism and mean motion resonance
- Kozai mechanism for crossing orbits
- Kozai mechanism for finite mass bodies