Simulation Astronomy

(Smoothed Particle Hydrodynamics)

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Smoothed Particle Hydrodynamics (SPH)

Gold and Monaghan 1977, Lucy 1977

SPH is a full Lagrangian method and is suitable to the following problems.

- Problems with a large dynamic range
  - Cosmological simulations, star formation simulations, . . .
- Problems with large deformation of objects with boundaries
  - Collision of objects (star and star, protoplanet and protoplanet,. . .)
  - fluid, elastic bodies. . .

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Advantage

- **No advection at the gas velocity.**
  - In mesh codes, advection causes a large amount of errors.
  - In SPH, particles move at v. You do not need to care about
- Easy to perform analyses based on the Lagrangian picture.

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Disadvantage (Compared with mesh codes)

- Large computational cost
  - For each particle, you need to calculate intercations among many particles.
- not good at treating field quantities that are important in low density regions.
  - **Iwasaki & Inutsuka (2011)** applies SPH to MHD by using Godunov method

Each Particle Has Density Profile

- Each particle is not point mass but is a sphere with a size of the smoothing length h
- The density profile of the i-particle is expressed in terms of the mass m_i and the kernel function $W(\boldsymbol{x} - \boldsymbol{x}_i, h)$.

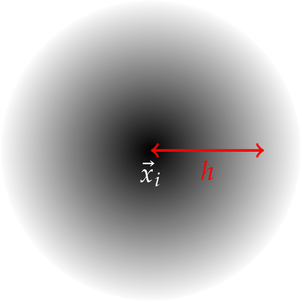
density distribution of the i-th particle

$$\rho_i(\boldsymbol{x}) = m_i W(\boldsymbol{x} - \boldsymbol{x}_i, h)$$

Conditions that W must meet

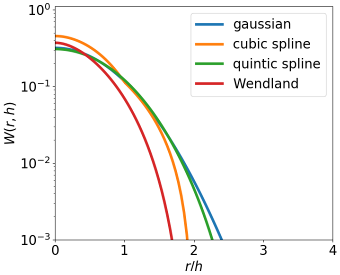
$$\lim_{h \to 0} W(\boldsymbol{x}, h) = \delta(\boldsymbol{x})$$

$$\int d^3x W(\boldsymbol{x}, h) = 1$$



Kernel Function

Kernel functions determine the numerical stability and the amount of errors.



- **Gaussian kernel.**
  It produces accurate results.
  Disadvantage: it extends to infinity.
  We neglect the contribution from $r \gtrsim 3h$

$$W(r,h) = \left(\frac{1}{\sqrt{\pi}h}\right)^D e^{-(r/h)^2}$$

- **a serise of spline kernels**
  The cubic spline kernel is one of the most commonly used kernel functions.
  It induces numerical instability in some situations and introduce a large amount of errors.
- **Wendland kernels** (Wendland 1995, Dehnen & Aly 2012)
  In SPH, pairing instability occurs when h is too large. It induces numerical clumping of SPH particles. Wendland kernels are stable for pairing instability. The Gaussian kernel is stable for this instaiblity only when an infinite extent is considered.

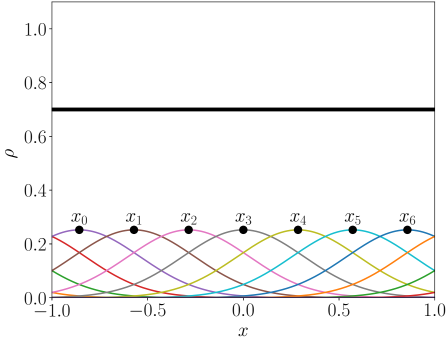
Expression of Density Profile by an Ensample of SPH Particles

Density profile is expressed by the summation of the density profiles of SPH particles.

$$\rho(\boldsymbol{x}) = \sum_i \rho_i(\boldsymbol{x}) = \sum_i m_i W(\boldsymbol{x} - \boldsymbol{x}_i, h)$$

Note that $\rho(\boldsymbol{x})$ is defined at any locations.

An Example of Density Profile



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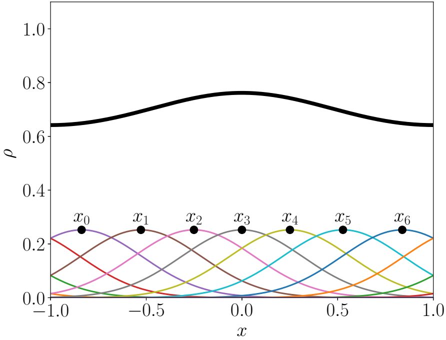

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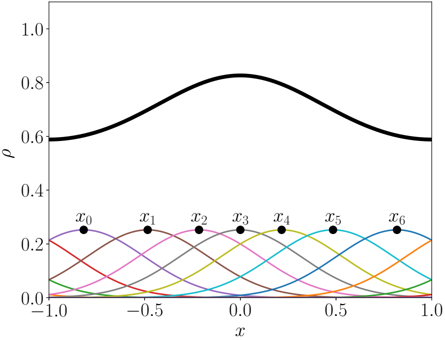

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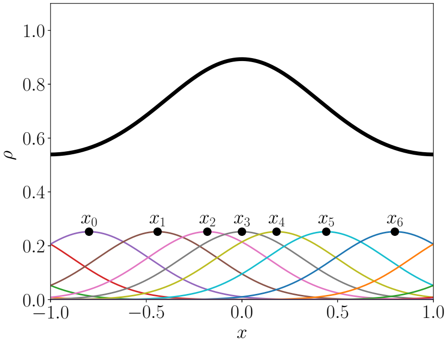

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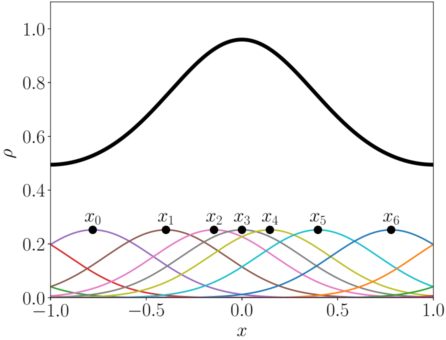

Definitions of Spatial Distributions of Other Physical Quantities

Given a quantity $f(\boldsymbol{x})$, the value of f at $\boldsymbol{x}$ is approximated by the convolution with W as follows:

$$\langle f\rangle(\boldsymbol{x}) \equiv \int d^3x' f(\boldsymbol{x}')W(\boldsymbol{x}-\boldsymbol{x}', h)$$

If $W(\boldsymbol{x}-\boldsymbol{x}', h)$ is $\delta(\boldsymbol{x}-\boldsymbol{x}')$, $\langle f\rangle(\boldsymbol{x}) \rightarrow f(\boldsymbol{x})$.

For SPH, the physical quantities are defined only at the positions of the SPH particles. You cannot integrate the above equation directly.

The space integral is approximated as the summation with respect to particles.
A volume of SPH particles is $\int d^3x' \sim \sum_j (m_j/\rho_j)$, $\boldsymbol{x}' \rightarrow \boldsymbol{x}_j$、 $f(\boldsymbol{x}') \rightarrow f_j$.

$$\langle f\rangle(\boldsymbol{x}) \equiv \underbrace{\int d^3x'}_{\sim\sum_j (m_j/\rho_j)} \underbrace{f(\boldsymbol{x}')W(\boldsymbol{x}-\boldsymbol{x}', h)}_{\boldsymbol{x}'\rightarrow \boldsymbol{x}_j}$$

$$f(\boldsymbol{x}) \sim \sum_j \frac{m_j}{\rho_j} f_j W(\boldsymbol{x}-\boldsymbol{x}_j, h),$$

$\langle\rangle$ is omitted.

Gradient in SPH

Using the spatial distribution $f(\boldsymbol{x})$ defined by the particle summation

$$f(\boldsymbol{x}) \sim \sum_j \frac{m_j}{\rho_j} f_j W(\boldsymbol{x} - \boldsymbol{x}_j, h),$$

the gradient of $f(\boldsymbol{x})$ at the particle position is calculated as follows:

$$\nabla f(\boldsymbol{x})\Big|_{\boldsymbol{x}=\boldsymbol{x}_i} \sim \sum_j \frac{m_j}{\rho_j} f_j \frac{\partial W(\boldsymbol{x} - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}}\Big|_{\boldsymbol{x}=\boldsymbol{x}_i} = \sum_j \frac{m_j}{\rho_j} f_j \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i}$$

There are other SPH expressions of the gradient.

Anti-symmetric and Symmetric Expressions of Gradient

Anti-symmetric Form

$$\nabla f(x)\Big|_{x=x_i} = \frac{1}{\rho}\left\{\nabla(\rho f(x)) - f(x)\nabla\rho\right\}\Big|_{x=x_i}$$

$$\nabla f(x)\Big|_{x=x_i} = \frac{1}{\rho}\left\{\sum_j \frac{m_j}{\rho_j}(\rho_j f_j)\nabla W(x - x_j, h) - f(x)\sum_j m_j \nabla W(x - x_j, h)\right\}\Big|_{x=x_i}$$

$$\nabla f(x)\Big|_{x=x_i} = \frac{1}{\rho_i}\sum_j m_j (f_j - f_i)\frac{\partial W(x_i - x_j, h)}{\partial x_i}$$

- When $f(x)$ is constant, the gradient becomes exactly zero.
- Often used to compute gradients.

Symmetric Form

$$\nabla f(x)\Big|_{x=x_i} = \rho\left\{\nabla\left(\frac{f(x)}{\rho}\right) + \frac{f(x)}{\rho^2}\nabla\rho\right\}\Big|_{x=x_i}$$

$$\nabla f(x)\Big|_{x=x_i} = \rho_i \sum_j m_j \left(\frac{f_i}{\rho_i^2} + \frac{f_j}{\rho_j^2}\right)\frac{\partial W(x_i - x_j, h)}{\partial x_i}$$

Accuracy of SPH Discretization

$$\langle f \rangle(\boldsymbol{x}) \equiv \int d^3x' f(\boldsymbol{x}')W(\boldsymbol{x} - \boldsymbol{x}', h)$$

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$$\langle f \rangle(\boldsymbol{x}) = \int d^3x' \left\{ f(\boldsymbol{x}) + \boldsymbol{\nabla} f(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{x'}) + O(|\boldsymbol{x} - \boldsymbol{x'}|^2) \right\} W(\boldsymbol{x} - \boldsymbol{x'}, h)$$

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$$\langle f\rangle(\boldsymbol{x}) \equiv \int d^3x' f(\boldsymbol{x}')W(\boldsymbol{x}-\boldsymbol{x}', h)$$

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$$\begin{aligned}\langle f\rangle(\boldsymbol{x}) &= f(\boldsymbol{x})\underbrace{\int d^3x' W(\boldsymbol{x}-\boldsymbol{x}', h)}_{=1} + \boldsymbol{\nabla} f(\boldsymbol{x})\cdot\underbrace{\int d^3x'(\boldsymbol{x}-\boldsymbol{x}')W(\boldsymbol{x}-\boldsymbol{x}', h)}_{=\mathbf{0}} + O(h^2)\\ &= f(\boldsymbol{x}) + O(h^2)\end{aligned}$$

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In reality,

$$\langle f \rangle(\boldsymbol{x}) \sim f(\boldsymbol{x}) \sum_i \underbrace{\frac{m_i}{\rho_i} W(\boldsymbol{x} - \boldsymbol{x}_i, h)}_{\neq 1} + \nabla f(\boldsymbol{x}) \cdot \underbrace{\sum_i \frac{m_i}{\rho_i} (\boldsymbol{x} - \boldsymbol{x}_i) W(\boldsymbol{x} - \boldsymbol{x}_i, h)}_{\neq 0} + O(h^2)$$

if the spatial distribution of particles is irregular.

Lagrangian Form of Hydrodynamic Equations

In Lagrangian picture, Hydrodynamic equations become

$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \quad \text{or} \quad \frac{d\,(1/\rho)}{dt} - \frac{1}{\rho}\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

$$\frac{d\boldsymbol{v}}{dt} + \frac{1}{\rho}\boldsymbol{\nabla} P = 0$$

$$\frac{de}{dt} + \frac{P}{\rho}\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \;\; \text{or} \;\; \frac{d}{dt}\left[\frac{v^2}{2} + e\right] + \frac{1}{\rho}\boldsymbol{\nabla} \cdot \left[\left(\frac{1}{2}v^2 + e\right)\boldsymbol{v}\right] = 0$$

Most SPH methods do not solve the continuity equation explicitly.

- The distribution of particles express the density

$$\rho_i = \sum_j m_j W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)$$

- In some cases, such as problems with boundaries, a continuous equation may be solved.

Equation of Motion of an SPH Particle

$$\frac{d\vec{v}_i}{dt} = -\frac{1}{\rho_i}\frac{\partial P(\mathrm{x})}{\partial \boldsymbol{x}}\Big|_{x=x_i}$$

If the anti-symmetric form is used,

$$\frac{\partial P(\mathbf{x})}{\partial \boldsymbol{x}}\Big|_{x=x_i} = \sum_j \frac{m_j}{\rho_j}(P_j - P_i)\frac{\partial}{\partial \boldsymbol{x}_i}W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)$$

Finally, we get

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j \frac{m_j}{\rho_i \rho_j}(P_j - P_i)\frac{\partial}{\partial \boldsymbol{x}_i}W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)$$

Conservation Law

- For finite-volume methods, the density, momentum, and the total energy are conserved within round-off errors.
- For SPH, the mass conservation is clearly satisfied,

$$M = \sum_i m_i$$

However, the momentum and energy conservations are not always satisfied.

Let Us Check Momentum Conservation

A formal form of the equation of motion of the i-th particle can be written as $m_i d\boldsymbol{v}_i/dt = \sum_j \boldsymbol{F}_{i\leftarrow j}$.

$\boldsymbol{F}_{i\leftarrow j}$ is the force exerted by particle j on particle i.

If the force $F_{i\leftarrow j}$ satisfies what conditions, momentum is conserved?

$$\begin{aligned}
\frac{d}{dt}\sum_i m_i \boldsymbol{v}_i &= \sum_i \sum_j \boldsymbol{F}_{i\leftarrow j} \\
&= \frac{1}{2}\sum_i \sum_j \boldsymbol{F}_{i\leftarrow j} + \frac{1}{2}\sum_i \sum_j \boldsymbol{F}_{i\leftarrow j} \\
&= \frac{1}{2}\sum_i \sum_j \boldsymbol{F}_{i\leftarrow j} + \frac{1}{2}\sum_j \sum_i \boldsymbol{F}_{j\leftarrow i} \\
&= \frac{1}{2}\sum_i \sum_j (\boldsymbol{F}_{i\leftarrow j} + \boldsymbol{F}_{j\leftarrow i})
\end{aligned}$$

If the action-reaction between particle i and particle j is satisfied, the momentum conservation is guaranteed.

$$\boldsymbol{F}_{i\leftarrow j} = -\boldsymbol{F}_{i\leftarrow j}$$

Let Us Check Momentum Conservation

$$m_i \frac{d\boldsymbol{v}_i}{dt} = -\sum_j \frac{m_i m_j}{\rho_i \rho_j} (P_j - P_i) \frac{\partial}{\partial \boldsymbol{x}_i} W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)$$

$$\boldsymbol{F}_{i \leftarrow j} = \frac{m_i m_j}{\rho_i \rho_j} (P_j - P_i) \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i}$$

$$\boldsymbol{F}_{j \leftarrow i} = \frac{m_j m_i}{\rho_j \rho_i} (P_i - P_j) \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_j} = \frac{m_j m_i}{\rho_j \rho_i} (P_j - P_i) \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i} = \boldsymbol{F}_{i \leftarrow j}$$

Clearly, the derived equation of motion does not satisfy action-reaction $\boldsymbol{F}_{i \leftarrow j} \neq -\boldsymbol{F}_{j \leftarrow i}$

$\Longrightarrow$ The total momentum is not conserved within round-off errors.

Equation of Motion of SPH Particles

Use the symmetric form of $\partial P / \partial \boldsymbol{x}$.

$$m_i \frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_i m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i}$$

This expression satisfies the momentum conservation.

Internal Energy Equation

In SPH, instead of the total energy, the internal energy is updated.

$$\frac{de_i}{dt} = -\frac{P}{\rho}\nabla \cdot v\bigg|_{x=x_i}$$

There are several expressions of the internal energy equation for SPH.
Most commonly used one is

$$\frac{de_i}{dt} = -\frac{P_i}{\rho_i^2}\sum_j m_j(\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i}$$

Energy Equation

Let us derive the total energy equation consistent with the internal energy equation shown in the previous slide.

$$\frac{dE_i}{dt} = \frac{de_i}{dt} + \frac{1}{2}\frac{d\boldsymbol{v}_i^2}{dt} = \frac{de_i}{dt} + \boldsymbol{v}_i \cdot \frac{d\boldsymbol{v}_i}{dt}$$

$$\frac{dE_i}{dt} = -\frac{P_i}{\rho_i^2}\sum_j m_j(\boldsymbol{v}_j - \boldsymbol{v}_i)\cdot\frac{\partial W}{\partial \boldsymbol{x}_i} - \boldsymbol{v}_i \cdot \sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)\frac{\partial W}{\partial \boldsymbol{x}_i}$$

$$\frac{dE_i}{dt} = \sum_j m_j \underbrace{\left(\frac{P_i}{\rho_i^2}\boldsymbol{v}_j + \frac{P_j}{\rho_j^2}\boldsymbol{v}_i\right)\frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i}}_{\text{anti-symmetric against } i \leftrightarrow j}$$

Note that using the internal energy equation does **NOT** guarantee the total energy conservation within round-off errors.

$$\frac{1}{\Delta t}\left\{\frac{(\boldsymbol{v}_i^{n+1})^2}{2} - \frac{(\boldsymbol{v}_i^n)^2}{2}\right\} = \left(\boldsymbol{v}_i^n + \frac{\Delta t}{2}\frac{d\boldsymbol{v}_i}{dt}\right)\cdot\frac{d\boldsymbol{v}_i}{dt} \neq \boldsymbol{v}_i^n \cdot \frac{d\boldsymbol{v}_i}{dt}$$

SPH Basic Equations

$$\rho_i = \sum_j m_j W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)$$

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i}$$

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They are traditional SPH equations.

Artificial Viscosity

To capture shock waves, dissipation is required also in SPH.

Unlike the mesh methods, artificial viscosity is often used in SPH.

Monaghan (1988) proposed the following artificial viscosity whose stress is a summation of the terms $\propto \nabla \cdot \boldsymbol{v}$ and $\propto (\nabla \cdot \boldsymbol{v})^2$

$$\Pi_{ij} = \begin{cases} \dfrac{-\alpha c_{ij}\mu_{ij} + \beta\mu_{ij}^2}{\rho_{ij}} & \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) \cdot \left(\boldsymbol{x}_i - \boldsymbol{x}_j\right) < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{ij} = \frac{h\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) \cdot \left(\boldsymbol{x}_i - \boldsymbol{x}_j\right)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^2} \sim h\nabla \cdot \boldsymbol{v}$$

α and β should be determined depending on problems that you are considering.
$\alpha = 1$ and $\beta = 2$ are often used.

Recent SPH uses an artificial viscosity based on Riemann problem (Monaghan 1997).

SPH Basic Equations

with Artificial Viscosity

$$\rho_i = \sum_j m_j W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)$$

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij} \right) \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i}$$

$$\frac{de_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{1}{2}\Pi_{ij} \right) (\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_i}$$

Variable Smoothing Length

Constant smoothing length works well to simulate fluid and elastic bodies with small density variations

However, in many astrophysical phenomena, the density changes significantly. Constant smoothing length is never used.

The mass of SPH particles is constant with time.
$\longrightarrow$ the size "h" is decreased as the density increases.

- **Smoothing length consistent with density**
  It is reasonable that h_i is determined so that m_i is equal to ρ_i multiplied by volume h_i^3.

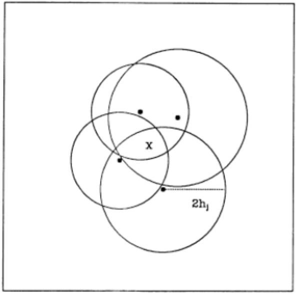
$$h_i = C_h \left(\frac{m_i}{\rho_i} \right)^{1/D}, \quad \rho_i = \sum_j m_j W \left(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i \right)$$

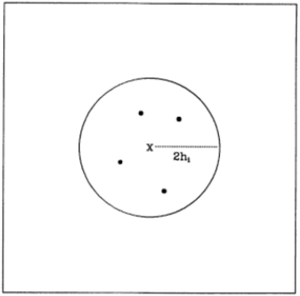
  C_h is a free parameter $\sim O(1)$ and D is the number of dimension.
  Because ρ_i depends on h_i, h_i and ρ_i are determined consistently by using an iterative method.

- **Fixed** N_{nb}
  h_i is determined so that the number of neighbor particles is fixed.

Estimation of Density with Variable h: Scatter versus Gather

$$\underbrace{\rho_i = \sum_j m_j W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_j)}_{\text{Scatter}} \quad \text{or} \quad \underbrace{\rho_i = \sum_j m_j W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i)}_{\text{Gather}}$$





Estimation of Density with Variable h:
Scatter versus Gather

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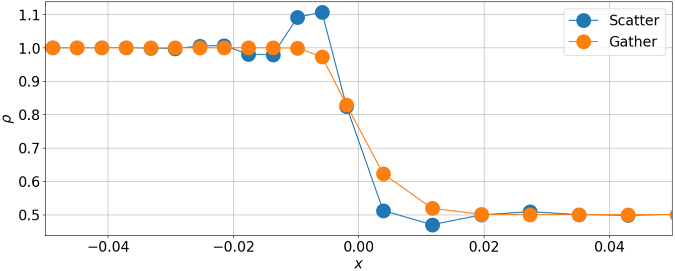
The particle separation in $x > 0$ is twice larger than that in $x < 0$.



Gather formula gives a monotonic profile.

$\Rightarrow$ Density is estimated by Gather formula

SPH Equations

with Variable h and Artificial Viscosity

$$\rho_i = \sum_j m_j W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i)$$

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_j \left\{ \left(\frac{P_i}{\rho_i^2} + \frac{\Pi_{ij}}{2} \right) \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i)}{\partial \boldsymbol{x}_i} + \left(\frac{P_j}{\rho_j^2} + \frac{\Pi_{ij}}{2} \right) \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_j)}{\partial \boldsymbol{x}_i}, \right\}$$

$$\frac{de_i}{dt} = \sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{\Pi_{ij}}{2} \right) \left(\boldsymbol{v}_i - \boldsymbol{v}_j \right) \cdot \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i)}{\partial \boldsymbol{x}_i},$$

The corresponding total energy equation

$$\frac{dE_i}{dt} = -\sum_j m_j \left\{ \left(\frac{P_i}{\rho_i^2} + \frac{\Pi_{ij}}{2} \right) \boldsymbol{v}_j \cdot \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i)}{\partial \boldsymbol{x}_i} + \left(\frac{P_j}{\rho_j^2} + \frac{\Pi_{ij}}{2} \right) \boldsymbol{v}_i \cdot \frac{\partial W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_j)}{\partial \boldsymbol{x}_i}, \right\}$$

CFL Condition

$$\Delta t = C_{\mathrm{CFL}} \min_{i,j} \left(\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{c_i + |\boldsymbol{v}_i - \boldsymbol{v}_j|} \right),$$

where $c_i = \sqrt{\gamma P_i / \rho_i}$ is the sound speed of the i-th particle. $|\boldsymbol{x}_i - \boldsymbol{x}_j|$ can be replaced by $\min(h_i, h_j)$.

Numerical Procedure

Initial condition: $\boldsymbol{x}_i^0$, ρ_i^0, v_i^0, P_i^0, e_i^0 ($\boldsymbol{x}_i^0$ and ρ_i^0 are determined consistently)

1. $(d\boldsymbol{v}_i/dt)^n$ is computed
2. $(de_i/dt)^n$ is computed
3. update the velocity

$$\boldsymbol{v}_i^{n+1} = \boldsymbol{v}_i^n + \left(\frac{d\boldsymbol{v}_i}{dt}\right)^n \Delta t$$

4. update the internal energy

$$e_i^{n+1} = e_i^n + \left(\frac{de_i}{dt}\right)^n \Delta t$$

5. update the particle position

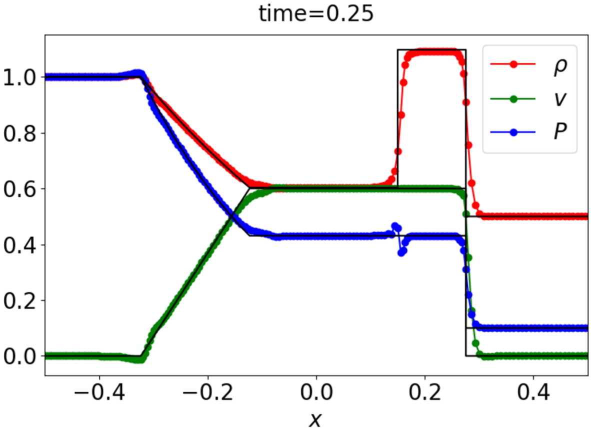
$$x_i^{n+1} = x_i^n + \frac{v_i^n + v_i^{n+1}}{2}\Delta t$$

6. update the density from the updated $\boldsymbol{x}_i$.
7. Pressure is updated $P_i^{n+1} = P(\rho_i^{n+1}, e_i^{n+1})$
8. Δt is determined. return to 1.

Sod Solution

the sample cod:

https://colab.research.google.com/drive/1pQtfqVeqU72E7K3Pnfpw1xlxkJ3mYj_6?usp=sharing

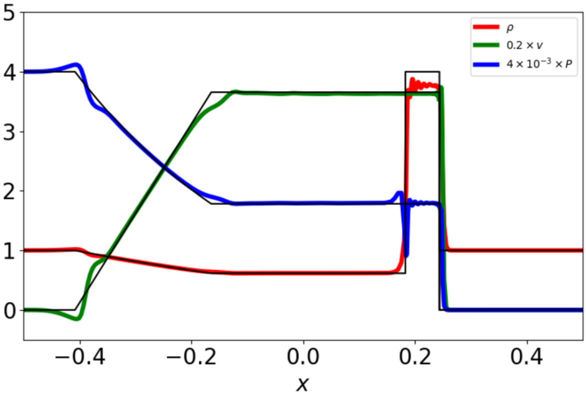


Problems

- The pressure has a numerical error around the CD (explained later).

Blast Wave

A more severe test problem ($P_{\mathrm{L}} = 10^3$, $P_{\mathrm{R}} = 0.01$, $\rho = 1$, $v = 0$).
the sample cod: `https://colab.research.google.com/drive/1AiN4yOByYM2t80Ss6zOQo5SkOkWViTuJ?usp=sharing`



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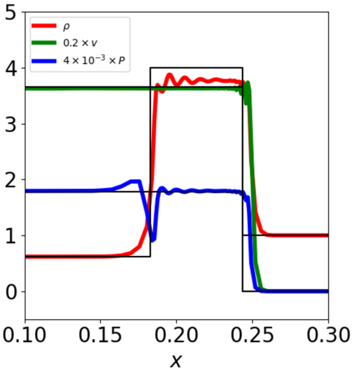

Problems

- The pressure has a numerical error around the CD (explained later).
- **Wrong shock jump condition and propagation speed of the shock front** because the total energy does not conserve within round-off errors.

Godunov SPH (Inutsuka 2002)

Inutsuka (2002) reformulated SPH and applied Godunov method into the reformulated SPH (Godunov SPH, GSPH).

Reformulate → keeps an integral form as much as possible.

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_j \int d^D x \frac{P(\boldsymbol{x})}{\rho(\boldsymbol{x})^2} \left(\frac{\partial W(\boldsymbol{x}-\boldsymbol{x}_i, h)}{\partial \boldsymbol{x}_i} W(\boldsymbol{x}-\boldsymbol{x}_j, h) - W(\boldsymbol{x}-\boldsymbol{x}_i, h) \frac{\partial W(\boldsymbol{x}-\boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_j} \right)$$

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Godunov method is applied into the interaction between the particles i and j.

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_j P^*_{ij} \int d^D x \frac{1}{\rho(\boldsymbol{x})^2} \left(\frac{\partial W(\boldsymbol{x}-\boldsymbol{x}_i, h)}{\partial \boldsymbol{x}_i} W(\boldsymbol{x}-\boldsymbol{x}_j, h) - W(\boldsymbol{x}-\boldsymbol{x}_i, h) \frac{\partial W(\boldsymbol{x}-\boldsymbol{x}_j, h)}{\partial \boldsymbol{x}_j} \right)$$

P^*_{ij} is the gas pressure evaluated from the result of the exact Riemann solver where the left state is the particle j and right state is the particle i.

Godunov SPH (Inutsuka 2002)

The simplest implementation is to approximate the kernel function as the delta function.

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_j P^*_{ij} \int d^D x \frac{1}{\rho(\boldsymbol{x})^2} \left(\frac{\partial W(\boldsymbol{x}-\boldsymbol{x}_i)}{\partial \boldsymbol{x}_i} \underbrace{W(\boldsymbol{x}-\boldsymbol{x}_j,h)}_{\to\delta(\boldsymbol{x}-\boldsymbol{x}_j)} - \underbrace{W(\boldsymbol{x}-\boldsymbol{x}_i,h)}_{\to\delta(\boldsymbol{x}-\boldsymbol{x}_i)} \frac{\partial W(\boldsymbol{x}-\boldsymbol{x}_j,h)}{\partial \boldsymbol{x}_j} \right)$$

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_j P^*_{ij} \left(\frac{1}{\rho_i^2} + \frac{1}{\rho_j^2} \right) \frac{\partial W(\boldsymbol{x}-\boldsymbol{x}_i,h)}{\partial \boldsymbol{x}_i}$$

Inutsuka (2002) approximates the integral part by using polynomial interpolation of $1/\rho(\boldsymbol{x})$ between the particles i and j.

Godunov SPH (Inutsuka 2002)

The simplest implementation of de_i/dt is as follows:

$$\frac{de_i}{dt} = -\sum_j m_j P^*_{ij} \left(\boldsymbol{v}^*_{ij} - \boldsymbol{v}_i - \frac{1}{2}\frac{d\boldsymbol{v}_i}{dt}\Delta t \right) \left(\frac{1}{\rho_i^2} + \frac{1}{\rho_j^2} \right) \frac{\partial W(\boldsymbol{x} - \boldsymbol{x}_i, h)}{\partial \boldsymbol{x}_i}$$

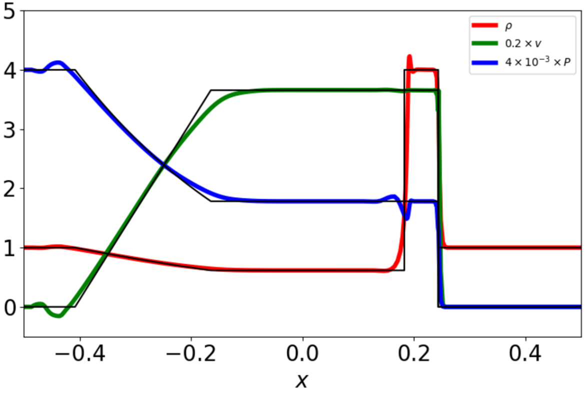
More sophisticated implementations are found in Inutsuka (2002).

Unlike the standard SPH, GSPH conserves the total energy within round-off errors.

$$\sum_j m_j \left\{ e_j^{n+1} + \frac{1}{2}\left(\boldsymbol{v}_j^{n+1}\right)^2 \right\} = \sum_j m_j \left\{ e_j^{n} + \frac{1}{2}\left(\boldsymbol{v}_j^{n}\right)^2 \right\}$$

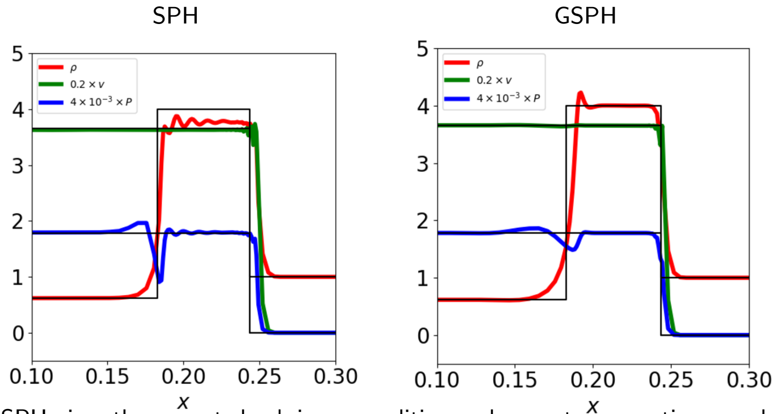
Blast Wave in GSPH

- The simple implementation of GSPH is used.
- The MUSCL method is to derive the initial condition of the Riemann problem (Inutsuka 2002, Iwasaki & Inutsuka 2011).



Blast Wave in GSPH

the sample cod: `https://colab.research.google.com/drive/1EI6hYGglWVHNs3j5KBzPhGUfQ12cLVOS?usp=sharing`



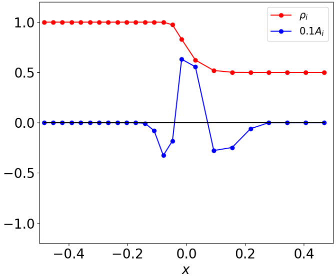
- GSPH gives the correct shock jump condition and correct propagation speed of the shock wave.
- Numerical oscillations seen in SPH disappears in GSPH.
- Pressure wiggles around the CD are decreased even in the simple implementation of GSPH.

Behaviors Around Contact Discontinuities (CDs)

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \frac{\partial}{\partial \boldsymbol{x}_i} W$$

The pressure is constant $P = P_0$ across the CD. The acceleration of the i-th particle should be zero.

$$\sum_j m_j P_0 \left(\frac{1}{\rho_i^2} + \frac{1}{\rho_j^2} \right) \frac{\partial W}{\partial \boldsymbol{x}_i} = 0 \quad \Longrightarrow \quad A_i = \sum_j m_j \left(\frac{1}{\rho_i^2} + \frac{1}{\rho_j^2} \right) \frac{\partial W}{\partial \boldsymbol{x}_i} = 0$$



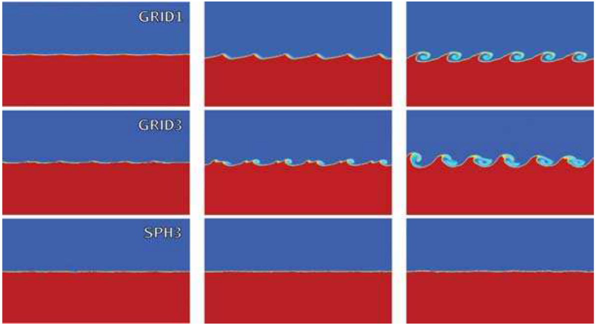
A_i is not equal to zero around the CD.

In simulations, P changes so that the acceleration around CDs becomes zero.

Artificial Tension Acting on Contact Discontinuities

(Agertz et al. 2007)

Artificial tension acts around contact discontinuities.
→ suppresses Kelvin-Helmholtz instability.

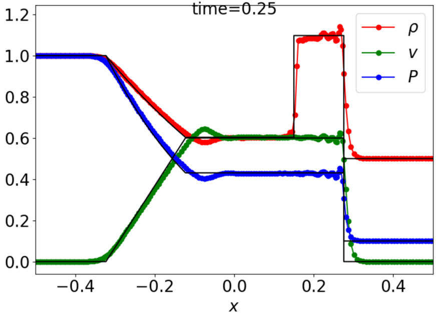


- This is a serious problem because the KH instability is an essential process in the generation of turbulence and mixing of different components.

One Possible Solution:

EOM Derived form Anti-symmetric Formula

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j \frac{m_j}{\rho_i \rho_j}(P_j - P_i)\frac{\partial}{\partial \boldsymbol{x}_i}W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i) - \sum_j m_j \Pi_{ij} \frac{1}{2}\left\{\frac{\partial}{\partial \boldsymbol{x}_i}W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i) + \frac{\partial}{\partial \boldsymbol{x}_i}W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_j)\right\}$$

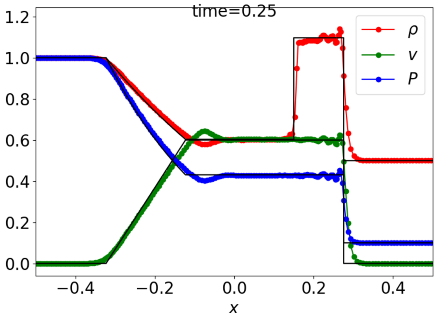


If the momentum conservation is allowed to be violated, pressure wiggles around CD disappear.

One Possible Solution:

EOM Derived form Anti-symmetric Formula

$$\frac{d\boldsymbol{v}_i}{dt} = -\sum_j \frac{m_j}{\rho_i \rho_j}(P_j - P_i)\frac{\partial}{\partial \boldsymbol{x}_i}W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i) - \sum_j m_j \Pi_{ij}\frac{1}{2}\left\{\frac{\partial}{\partial \boldsymbol{x}_i}W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i) + \frac{\partial}{\partial \boldsymbol{x}_i}W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_j)\right\}$$



If the momentum conservation is allowed to be violated, pressure wiggles around CD disappear.

BUT, the violation of the momentum conservation can cause another problems.

$\Rightarrow$ different approaches have been proposed

Methods to Reduce Artificial Surface Tension

Keep using the symmetric form of EOM.

- **Artificial Thermal Conduction** (Price 2007)
  - It makes the internal energy smooth around CDs
- **Godunov SPH** (Inutsuka 2002, Cha et al. 2010)
  - The numerical error in A_i is decreased by appropriately approximating an integral form of A_i.
  - Riemann solver introduces thermal conduction.
- **Density-Independent SPH** (Saitoh & Makino 2012)
  - As the volme of particles, DISPH adopts $(\gamma - 1) m_i e_i / P_i$ (also see Ritchie & Thomas 2001) instead of m_i/ρ_i.
  - it does not use the gas density explicitly.

- All methods can solve the development of the KH instability.
- BUT, most authors discuss only appearance of vortices, but neither method is completely error-free. Convergence to an exact solution for the growth rate is very slow (McNally et al. 2012,Tricco 2018)。
  - For SPH with artificial heat conduction, the kernel must be a higher order spline to obtained converged results (Tricco 2018).