Jacobi 行列を用いた二体問題関係変数の偏微分係数

以下の議論では $a, e, \omega, I, \Omega, l$がいわゆるケプラー 6 軌道要素（軌道半長径、離心率、近点引数、軌道傾斜角、昇交点黄経、平均近点離角）を表し、u が離心近点離角、fが真近点離角を表わす。正準変数としては Delauney 変数 L, G, H, l, g, h を用い、それぞれ以下のような意味を持つとする。

$$\begin{aligned} L &= \sqrt{\mu a}, & l &= l, \\ G &= L\sqrt{1-e^2}, & g &= \omega, \\ H &= G\cos I, & h &= \Omega. \end{aligned} \tag{1}$$

また、

$$\eta \equiv \sqrt{1-e^2}, \tag{2}$$

$$r = a(1 - e\cos u) = \frac{a(1-e^2)}{1+e\cos f}, \tag{3}$$

であり、離心近点離角 u と真近点離角 fの関係式である以下を頻用する。

$$\sin u = \frac{\eta \sin f}{1+e\cos f}, \quad \cos u = \frac{e+\cos f}{1+e\cos f}, \tag{4}$$

$$\sin f = \frac{\eta \sin u}{1-e\cos u}, \quad \cos f = \frac{\cos u - e}{1-e\cos u}. \tag{5}$$

ここでは、微分変数変換を Jacobian の形式で表現することにする。軌道運動を表わす独立六変数としては、直交座標 $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ の他に以下のような種類が考えられる。

$(a, e, \omega, I, \Omega, l)$ $\cdots$ いわゆる軌道 6 要素
$(a, e, \omega, I, \Omega, f)$ $\cdots$ 平均近点離角 lの代わりに真近点離角 fを用いる
$(a, e, \omega, I, \Omega, u)$ $\cdots$ 平均近点離角 lの代わりに離心近点離角 u を用いる
(L, G, H, l, g, h) $\cdots$ Delauney 要素

或る変数の組から或る変数の組への微分変換のやり方は${}_4P_2 = 12$ 通りある。

$$\begin{aligned}
(da, de, d\omega, dI, d\Omega, df) &\to (da, de, d\omega, dI, d\Omega, du) \\
(da, de, d\omega, dI, d\Omega, df) &\to (da, de, d\omega, dI, d\Omega, dl) \\
(da, de, d\omega, dI, d\Omega, df) &\to (dL, dG, dH, dl, dg, dh) \\
(da, de, d\omega, dI, d\Omega, du) &\to (da, de, d\omega, dI, d\Omega, dl) \\
(da, de, d\omega, dI, d\Omega, du) &\to (da, de, d\omega, dI, d\Omega, df) \\
(da, de, d\omega, dI, d\Omega, du) &\to (dL, dG, dH, dl, dg, dh) \\
(da, de, d\omega, dI, d\Omega, dl) &\to (da, de, d\omega, dI, d\Omega, df) \\
(da, de, d\omega, dI, d\Omega, dl) &\to (da, de, d\omega, dI, d\Omega, du) \\
(da, de, d\omega, dI, d\Omega, dl) &\to (dL, dG, dH, dl, dg, dh) \\
(dL, dG, dH, dl, dg, dh) &\to (da, de, d\omega, dI, d\Omega, dl) \\
(dL, dG, dH, dl, dg, dh) &\to (da, de, d\omega, dI, d\Omega, du) \\
(dL, dG, dH, dl, dg, dh) &\to (da, de, d\omega, dI, d\Omega, df)
\end{aligned}$$

以下ではこれらを順に追って列挙して行く。なお、本稿で使用する Jacobi 行列は通常の教科書にあるものと行と列の意味が入れ替わっていることに注意する。

以下ではまず前半部として

$$
\begin{array}{c}
(da, de, d\omega, dI, d\Omega, df) \\
\downarrow \\
(da, de, d\omega, dI, d\Omega, du) \\
\downarrow \\
(da, de, d\omega, dI, d\Omega, dl) \\
\downarrow \\
(dL, dG, dH, dl, dg, dh)
\end{array}
$$

の方向の変換について計算を行う。

$\boxed{(da, de, d\omega, dI, d\Omega, df) \to (da, de, d\omega, dI, d\Omega, du)}$

$$
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} = \frac{\partial(a, e, \omega, I, \Omega, f)}{\partial(a, e, \omega, I, \Omega, u)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{6}
$$

$$
= \begin{pmatrix}
\frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial u} \\
\frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial u} \\
\frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial u} \\
\frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial u} \\
\frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial u} \\
\frac{\partial f}{\partial a} & \frac{\partial f}{\partial e} & \frac{\partial f}{\partial \omega} & \frac{\partial f}{\partial I} & \frac{\partial f}{\partial \Omega} & \frac{\partial f}{\partial u}
\end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{7}
$$

$$
= \begin{pmatrix}
1 & & & \mathbf{0} & & 0 \\
& 1 & & & & 0 \\
& & 1 & & & 0 \\
& & & 1 & & 0 \\
\mathbf{0} & & & & 1 & 0 \\
0 & \frac{\partial f}{\partial e} & 0 & 0 & 0 & \frac{\partial f}{\partial u}
\end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{8}
$$

式 (8) の変換行列に於いては $a, e, \omega, I, \Omega, u$ はすべて独立であるから

$$
\frac{\partial u}{\partial e} = 0, \quad \frac{\partial e}{\partial u} = 0 \tag{9}
$$

これを用いて式 (5) の第二式を e で偏微分すると

$$
\frac{\partial}{\partial e} \cos f = -\sin f \frac{\partial f}{\partial e} = \frac{\partial}{\partial e} \Big((\cos u - e)(1 - c \cos u)^{-1} \Big) \tag{10}
$$

$$
= -1 \cdot (1 - e \cos u)^{-1} + (-e + \cos u) \cdot (-1) \cdot (1 - e \cos u)^{-2} (-\cos u) \tag{11}
$$

$$
= \frac{-1}{1 - e \cos u} + \frac{-e + \cos u}{(1 - e \cos u)^2} \cos u \tag{12}
$$

$$= \frac{1}{(1-e\cos u)^2}\left[-1+e\cos u+(-e+\cos u)\cos u\right] \tag{13}$$

$$= \frac{1}{(1-e\cos u)^2}\left(\cos^2 u-1\right) \tag{14}$$

$$= \frac{-\sin^2 u}{(1-e\cos u)^2} \tag{15}$$

$$= -\left(\frac{\sin u}{1-e\cos u}\right)^2 \tag{16}$$

$$= -\left(\frac{\sin f}{\eta}\right)^2 \quad (\because\ (4)) \tag{17}$$

$$\therefore\ \frac{\partial f}{\partial e} = \left(\frac{1}{-\sin f}\right)\left(-\frac{\sin^2 f}{\eta^2}\right) = \frac{1}{\eta^2}\sin f = \frac{L^2}{G^2}\sin f \tag{18}$$

同様にして式 (5) の第二式を u で偏微分すると

$$\frac{\partial}{\partial u}\cos f = -\sin f\frac{\partial f}{\partial u} = \frac{\partial}{\partial u}\left((\cos u-e)(1-c\cos u)^{-1}\right) \tag{19}$$

$$= -\sin u(1-e\cos u)^{-1}+(-e+\cos u)\cdot(-1)\cdot(1-e\cos u)^{-2}e\sin u \tag{20}$$

$$= \frac{-\sin u}{1-e\cos u}-\frac{-e+\cos u}{(1-e\cos u)^2}e\sin u \tag{21}$$

$$= \frac{1}{(1-e\cos u)^2}\left[-\sin u(1-e\cos u)-e\sin u(-e+\cos u)\right] \tag{22}$$

$$= \frac{\sin u}{(1-e\cos u)^2}(1-e^2) \tag{23}$$

$$= \frac{\eta^2\sin^2 u}{(1-e\cos u)^2}\left(-\frac{1}{\sin u}\right) \tag{24}$$

$$= -\frac{\sin^2 f}{\sin u} \quad (\because\ (4)) \tag{25}$$

$$\therefore\ \frac{\partial f}{\partial u} = \left(-\frac{1}{\sin f}\right)\left(-\frac{\sin^2 f}{\sin u}\right) = \frac{\eta\sin u}{1-e\cos u}\frac{1}{\sin u} = \frac{a\eta}{r} \quad (\because\ (3)) \tag{26}$$

以上より、式 (8) は以下のようになる。

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} = \frac{\partial(a,e,\omega,I,\Omega,f)}{\partial(a,e,\omega,I,\Omega,u)}\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \begin{pmatrix} 1 & & & & \text{\huge 0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \text{\huge 0} & & & & 1 & 0 \\ 0 & \frac{\partial f}{\partial e} & 0 & 0 & 0 & \frac{\partial f}{\partial u} \end{pmatrix}\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} 1 & & & & \text{\huge 0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \text{\huge 0} & & & & 1 & 0 \\ 0 & \frac{L^2}{G^2}\sin f & 0 & 0 & 0 & \frac{a\eta}{r} \end{pmatrix}\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{28}$$

$(da, de, d\omega, dI, d\Omega, du) \to (da, de, d\omega, dI, d\Omega, dl)$

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,l)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{29}$$

$$= \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial l} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial l} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial l} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial l} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial l} \\ \frac{\partial u}{\partial a} & \frac{\partial u}{\partial e} & \frac{\partial u}{\partial \omega} & \frac{\partial u}{\partial I} & \frac{\partial u}{\partial \Omega} & \frac{\partial u}{\partial l} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial u}{\partial e} & 0 & 0 & 0 & \frac{\partial u}{\partial l} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{31}$$

式 (31) の変換行列に於いては $a, e, \omega, I, \Omega, l$はすべて独立であるから

$$\frac{\partial l}{\partial e} = 0, \quad \frac{\partial e}{\partial l} = 0 \tag{32}$$

これを用いてケプラー方程式 Kepler 方程式

$$u - e \sin u = l \tag{33}$$

の両辺を e で偏微分すると、

$$\frac{\partial u}{\partial e} - \left(\sin u + e \cos u \frac{\partial u}{\partial e} \right) = \frac{\partial l}{\partial e} = 0 \tag{34}$$

$$\therefore \ (1 - e \cos u) \frac{\partial u}{\partial e} = \sin u \tag{35}$$

$$\therefore \ \frac{\partial u}{\partial e} = \frac{\sin u}{1 - e \cos u} = \frac{\sin f}{\eta} \quad (\because (5)) \tag{36}$$

同様に式 (33) の両辺を lで偏微分すると

$$\frac{\partial u}{\partial l} - e \cos \frac{\partial u}{\partial l} = \frac{\partial l}{\partial l} = 1 \tag{37}$$

$$\therefore \ (1 - e \cos u) \frac{\partial u}{\partial l} = 1 \tag{38}$$

$$\therefore \ \frac{\partial u}{\partial l} = \frac{1}{1 - e \cos u} = \frac{a}{r} \tag{39}$$

以上より、式 (31) は以下のようになる。

$$
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,l)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{40}
$$

$$
= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial u}{\partial e} & 0 & 0 & 0 & \frac{\partial u}{\partial l} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{41}
$$

$$
= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\sin f}{\eta} & 0 & 0 & 0 & \frac{a}{r} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{42}
$$

$(da, de, d\omega, dI, d\Omega, df) \rightarrow (da, de, d\omega, dI, d\Omega, dl)$

この変換は $df \rightarrow du \rightarrow dl$ と経由して行けば良く、

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} = \frac{\partial(a,e,\omega,I,\Omega,f)}{\partial(a,e,\omega,I,\Omega,l)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial l} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial l} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial l} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial l} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial l} \\ \frac{\partial f}{\partial a} & \frac{\partial f}{\partial e} & \frac{\partial f}{\partial \omega} & \frac{\partial f}{\partial I} & \frac{\partial f}{\partial \Omega} & \frac{\partial f}{\partial l} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{43}$$

$$= \frac{\partial(a,e,\omega,I,\Omega,f)}{\partial(a,e,\omega,I,\Omega,u)} \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,l)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{44}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial f}{\partial e} & 0 & 0 & 0 & \frac{\partial f}{\partial u} \end{pmatrix} \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial u}{\partial e} & 0 & 0 & 0 & \frac{\partial u}{\partial l} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{45}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{L^2}{G^2}\sin f & 0 & 0 & 0 & \frac{a\eta}{r} \end{pmatrix} \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\sin f}{\eta} & 0 & 0 & 0 & \frac{a}{r} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{46}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{L^2}{G^2}\sin f + \frac{a\eta}{r}\frac{\sin f}{\eta} & 0 & 0 & 0 & \frac{a\eta}{r}\frac{a}{r} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{47}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f & 0 & 0 & 0 & \frac{a^2\eta}{r^2} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{48}$$

$(da, de, d\omega, dI, d\Omega, dl) \to (dL, dG, dH, dl, dg, dh)$

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} = \frac{\partial(a, e, \omega, I, \Omega, l)}{\partial(L, G, H, l, g, h)} \begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{49}$$

$$= \begin{pmatrix} \frac{\partial a}{\partial L} & \frac{\partial a}{\partial G} & \frac{\partial a}{\partial H} & \frac{\partial a}{\partial l} & \frac{\partial a}{\partial g} & \frac{\partial a}{\partial h} \\ \frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & \frac{\partial e}{\partial H} & \frac{\partial e}{\partial l} & \frac{\partial e}{\partial g} & \frac{\partial e}{\partial h} \\ \frac{\partial \omega}{\partial L} & \frac{\partial \omega}{\partial G} & \frac{\partial \omega}{\partial H} & \frac{\partial \omega}{\partial l} & \frac{\partial \omega}{\partial g} & \frac{\partial \omega}{\partial h} \\ \frac{\partial I}{\partial L} & \frac{\partial I}{\partial G} & \frac{\partial I}{\partial H} & \frac{\partial I}{\partial l} & \frac{\partial I}{\partial g} & \frac{\partial I}{\partial h} \\ \frac{\partial \Omega}{\partial L} & \frac{\partial \Omega}{\partial G} & \frac{\partial \Omega}{\partial H} & \frac{\partial \Omega}{\partial l} & \frac{\partial \Omega}{\partial g} & \frac{\partial \Omega}{\partial h} \\ \frac{\partial l}{\partial L} & \frac{\partial l}{\partial G} & \frac{\partial l}{\partial H} & \frac{\partial l}{\partial l} & \frac{\partial l}{\partial g} & \frac{\partial l}{\partial h} \end{pmatrix} \begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{50}$$

L の定義を用いて a を Delauney 要素で表すと

$$a = \frac{L^2}{\mu} \tag{51}$$

なので、a は L にのみ依存することがわかる。従って

$$\frac{\partial a}{\partial L} = \frac{2L}{\mu}, \tag{52}$$

$$\frac{\partial a}{\partial G} = \frac{\partial a}{\partial H} = \frac{\partial a}{\partial l} = \frac{\partial a}{\partial g} = \frac{\partial a}{\partial h} = 0. \tag{53}$$

同様に、L, G の定義を用いて e を正準変数で表すと

$$\frac{G^2}{L^2} = 1 - e^2 \tag{54}$$

$$\therefore\ e = \sqrt{1 - \frac{G^2}{L^2}} \tag{55}$$

なので、e は L, G にのみ依存することがわかる。従って

$$\frac{\partial e}{\partial L} = \frac{\partial}{\partial L}\sqrt{1 - \frac{G^2}{L^2}} = \frac{1}{2}\left(1 - \frac{G^2}{L^2}\right)^{-\frac{1}{2}} \frac{2G^2}{L^3} = \frac{G^2}{eL^3}, \tag{56}$$

$$\frac{\partial e}{\partial G} = \frac{\partial}{\partial G}\sqrt{1 - \frac{G^2}{L^2}} = \frac{1}{2}\left(1 - \frac{G^2}{L^2}\right)^{-\frac{1}{2}} \left(-\frac{2G}{L^2}\right) = -\frac{G}{eL^2}, \tag{57}$$

$$\frac{\partial e}{\partial H} = \frac{\partial e}{\partial l} = \frac{\partial e}{\partial g} = \frac{\partial e}{\partial h} = 0. \tag{58}$$

また $g = \omega$ であるからして、g は他のすべての軌道要素 (a, e, I, Ω, l) に対して独立。従って

$$\frac{\partial g}{\partial L} = \frac{\partial g}{\partial G} = \frac{\partial g}{\partial H} = \frac{\partial g}{\partial l} = \frac{\partial g}{\partial h} = 0. \tag{59}$$

またも同様に、L, G, Hの定義を用いて Iを正準変数で表すと

$$\cos I = \frac{H}{G} \tag{60}$$

なので、Iは G, Hにのみ依存することがわかる。(60) の両辺を G で偏微分すると

$$-\frac{H}{G^2} = -\sin I \frac{\partial I}{\partial G}, \tag{61}$$

$$\therefore \frac{\partial I}{\partial G} = \frac{H}{G^2 \sin I} = \frac{G \cos I}{G^2 \sin I} = \frac{1}{G \tan I}. \tag{62}$$

同様に (60) の両辺を Hで偏微分すると

$$\frac{1}{G} = -\sin I \frac{\partial I}{\partial H}, \tag{63}$$

$$\therefore \frac{\partial I}{\partial H} = -\frac{1}{G \sin I}. \tag{64}$$

他の変数からはすべて独立なので

$$\frac{\partial I}{\partial L} = \frac{\partial I}{\partial l} = \frac{\partial I}{\partial g} = \frac{\partial I}{\partial h} = 0. \tag{65}$$

となる。従って式 (50) の変換行列は以下のようになる。

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{\partial I}{\partial G} & \frac{\partial H}{\partial G} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{66}$$

$$= \begin{pmatrix} \frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\ \frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{G \tan I} & -\frac{1}{G \sin I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{67}$$

$(da, de, d\omega, dI, d\Omega, df) \rightarrow (dL, dG, dH, dl, dg, dh)$

この変換は $df \rightarrow du \rightarrow dl \rightarrow dL$ と経由して行けば良く、

$$
\begin{aligned}
\frac{\partial(a,e,\omega,I,\Omega,f)}{\partial(L,G,H,l,g,h)} &= \frac{\partial(a,e,\omega,I,\Omega,f)}{\partial(a,e,\omega,I,\Omega,u)} \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,l)} \frac{\partial(a,e,\omega,I,\Omega,l)}{\partial(L,G,H,l,g,h)} && (68) \\
&= \begin{pmatrix}
\frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial u} \\
\frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial u} \\
\frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial u} \\
\frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial u} \\
\frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial u} \\
\frac{\partial f}{\partial a} & \frac{\partial f}{\partial e} & \frac{\partial f}{\partial \omega} & \frac{\partial f}{\partial I} & \frac{\partial f}{\partial \Omega} & \frac{\partial f}{\partial u}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial l} \\
\frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial l} \\
\frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial l} \\
\frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial l} \\
\frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial l} \\
\frac{\partial u}{\partial a} & \frac{\partial u}{\partial e} & \frac{\partial u}{\partial \omega} & \frac{\partial u}{\partial I} & \frac{\partial u}{\partial \Omega} & \frac{\partial u}{\partial l}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial a}{\partial L} & \frac{\partial a}{\partial G} & \frac{\partial a}{\partial H} & \frac{\partial a}{\partial l} & \frac{\partial a}{\partial g} & \frac{\partial a}{\partial h} \\
\frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & \frac{\partial e}{\partial H} & \frac{\partial e}{\partial l} & \frac{\partial e}{\partial g} & \frac{\partial e}{\partial h} \\
\frac{\partial \omega}{\partial L} & \frac{\partial \omega}{\partial G} & \frac{\partial \omega}{\partial H} & \frac{\partial \omega}{\partial l} & \frac{\partial \omega}{\partial g} & \frac{\partial \omega}{\partial h} \\
\frac{\partial I}{\partial L} & \frac{\partial I}{\partial G} & \frac{\partial I}{\partial H} & \frac{\partial I}{\partial l} & \frac{\partial I}{\partial g} & \frac{\partial I}{\partial h} \\
\frac{\partial \Omega}{\partial L} & \frac{\partial \Omega}{\partial G} & \frac{\partial \Omega}{\partial H} & \frac{\partial \Omega}{\partial l} & \frac{\partial \Omega}{\partial g} & \frac{\partial \Omega}{\partial h} \\
\frac{\partial l}{\partial L} & \frac{\partial l}{\partial G} & \frac{\partial l}{\partial H} & \frac{\partial l}{\partial l} & \frac{\partial l}{\partial g} & \frac{\partial l}{\partial h}
\end{pmatrix} \\
&= \begin{pmatrix}
1 & & & & \mathbf{0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\mathbf{0} & & & & 1 & 0 \\
0 & \frac{\partial f}{\partial e} & 0 & 0 & 0 & \frac{\partial f}{\partial u}
\end{pmatrix}
\begin{pmatrix}
1 & & & & \mathbf{0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\mathbf{0} & & & & 1 & 0 \\
0 & \frac{\partial u}{\partial e} & 0 & 0 & 0 & \frac{\partial u}{\partial u}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial a}{\partial L} & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{\partial I}{\partial G} & \frac{\partial H}{\partial G} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} && (69) \\
&= \begin{pmatrix}
1 & & & & \mathbf{0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\mathbf{0} & & & & 1 & 0 \\
0 & \frac{L^2}{G^2}\sin f & 0 & 0 & 0 & \frac{a\eta}{r}
\end{pmatrix}
\begin{pmatrix}
1 & & & & \mathbf{0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\mathbf{0} & & & & 1 & 0 \\
0 & \frac{\sin f}{\eta} & 0 & 0 & 0 & \frac{a}{r}
\end{pmatrix}
\begin{pmatrix}
\frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\
\frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{G\tan I} & -\frac{1}{G\sin I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} && (70) \\
&= \begin{pmatrix}
1 & & & & \mathbf{0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\mathbf{0} & & & & 1 & 0 \\
0 & \left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f & 0 & 0 & 0 & \frac{a^2\eta}{r^2}
\end{pmatrix}
\begin{pmatrix}
\frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\
\frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{G\tan I} & -\frac{1}{G\sin I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} && (71) \\
&= \begin{pmatrix}
\frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\
\frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{G\tan I} & -\frac{1}{G\sin I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{G^2}{eL^3}\left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f & -\frac{G}{eL^2}\left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f & 0 & \frac{a^2\eta}{r^2} & 0 & 0
\end{pmatrix} && (72) \\
&= && (73)
\end{aligned}
$$

即ち、

$$
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} = \frac{\partial(a,e,\omega,I,\Omega,f)}{\partial(L,G,H,l,g,h)} \begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{74}
$$

$$
= \begin{pmatrix}
\frac{\partial a}{\partial L} & \frac{\partial a}{\partial G} & \frac{\partial a}{\partial H} & \frac{\partial a}{\partial l} & \frac{\partial a}{\partial g} & \frac{\partial a}{\partial h} \\
\frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & \frac{\partial e}{\partial H} & \frac{\partial e}{\partial l} & \frac{\partial e}{\partial g} & \frac{\partial e}{\partial h} \\
\frac{\partial \omega}{\partial L} & \frac{\partial \omega}{\partial G} & \frac{\partial \omega}{\partial H} & \frac{\partial \omega}{\partial l} & \frac{\partial \omega}{\partial g} & \frac{\partial \omega}{\partial h} \\
\frac{\partial I}{\partial L} & \frac{\partial I}{\partial G} & \frac{\partial I}{\partial H} & \frac{\partial I}{\partial l} & \frac{\partial I}{\partial g} & \frac{\partial I}{\partial h} \\
\frac{\partial \Omega}{\partial L} & \frac{\partial \Omega}{\partial G} & \frac{\partial \Omega}{\partial H} & \frac{\partial \Omega}{\partial l} & \frac{\partial \Omega}{\partial g} & \frac{\partial \Omega}{\partial h} \\
\frac{\partial f}{\partial L} & \frac{\partial f}{\partial G} & \frac{\partial f}{\partial H} & \frac{\partial f}{\partial l} & \frac{\partial f}{\partial g} & \frac{\partial f}{\partial h}
\end{pmatrix}
\begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{75}
$$

$$
= \begin{pmatrix}
\frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\
\frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{G \tan I} & -\frac{1}{G \sin I} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{G^2}{eL^3}\left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f & -\frac{G}{eL^2}\left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f & 0 & \frac{a^2\eta}{r^2} & 0 & 0
\end{pmatrix}
\begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{76}
$$

二体問題に於ける symplectic integrator の誤差評価では上述の$\frac{\partial f}{\partial L}, \frac{\partial f}{\partial G}$などを頻用する。

$$\boxed{(da, de, d\omega, dI, d\Omega, du) \rightarrow (dL, dG, dH, dl, dg, dh)}$$

この変換は $du \rightarrow dl \rightarrow dL$ と経由して行けば良く、

$$\frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(L,G,H,l,g,h)} = \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,l)} \frac{\partial(a,e,\omega,I,\Omega,l)}{\partial(L,G,H,l,g,h)} \tag{77}$$

$$= \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial l} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial l} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial l} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial l} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial l} \\ \frac{\partial u}{\partial a} & \frac{\partial u}{\partial e} & \frac{\partial u}{\partial \omega} & \frac{\partial u}{\partial I} & \frac{\partial u}{\partial \Omega} & \frac{\partial u}{\partial l} \end{pmatrix} \begin{pmatrix} \frac{\partial a}{\partial L} & \frac{\partial a}{\partial G} & \frac{\partial a}{\partial H} & \frac{\partial a}{\partial l} & \frac{\partial a}{\partial g} & \frac{\partial a}{\partial h} \\ \frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & \frac{\partial e}{\partial H} & \frac{\partial e}{\partial l} & \frac{\partial e}{\partial g} & \frac{\partial e}{\partial h} \\ \frac{\partial \omega}{\partial L} & \frac{\partial \omega}{\partial G} & \frac{\partial \omega}{\partial H} & \frac{\partial \omega}{\partial l} & \frac{\partial \omega}{\partial g} & \frac{\partial \omega}{\partial h} \\ \frac{\partial I}{\partial L} & \frac{\partial I}{\partial G} & \frac{\partial I}{\partial H} & \frac{\partial I}{\partial l} & \frac{\partial I}{\partial g} & \frac{\partial I}{\partial h} \\ \frac{\partial \Omega}{\partial L} & \frac{\partial \Omega}{\partial G} & \frac{\partial \Omega}{\partial H} & \frac{\partial \Omega}{\partial l} & \frac{\partial \Omega}{\partial g} & \frac{\partial \Omega}{\partial h} \\ \frac{\partial l}{\partial L} & \frac{\partial l}{\partial G} & \frac{\partial l}{\partial H} & \frac{\partial l}{\partial l} & \frac{\partial l}{\partial g} & \frac{\partial l}{\partial h} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial u}{\partial e} & 0 & 0 & 0 & \frac{\partial u}{\partial u} \end{pmatrix} \begin{pmatrix} \frac{\partial a}{\partial L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{\partial I}{\partial G} & \frac{\partial H}{\partial G} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{78}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\sin f}{\eta} & 0 & 0 & 0 & \frac{a}{r} \end{pmatrix} \begin{pmatrix} \frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\ \frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{G\tan I} & -\frac{1}{G\sin I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{79}$$

$$= \begin{pmatrix} \frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\ \frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{G\tan I} & -\frac{1}{G\sin I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{G^2}{eL^3}\frac{\sin f}{\eta} & -\frac{G}{eL^2}\frac{\sin f}{\eta} & 0 & \frac{a}{r} & 0 & 0 \end{pmatrix} \tag{80}$$

$$= \begin{pmatrix} \frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\ \frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{G\tan I} & -\frac{1}{G\sin I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\eta}{eL}\sin f & -\frac{\sin f}{eL} & 0 & \frac{a}{r} & 0 & 0 \end{pmatrix} \tag{81}$$

即ち、

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \frac{\partial(a, e, \omega, I, \Omega, u)}{\partial(L, G, H, l, g, h)} \begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{82}$$

$$= \begin{pmatrix} \frac{\partial a}{\partial L} & \frac{\partial a}{\partial G} & \frac{\partial a}{\partial H} & \frac{\partial a}{\partial l} & \frac{\partial a}{\partial g} & \frac{\partial a}{\partial h} \\ \frac{\partial e}{\partial L} & \frac{\partial e}{\partial G} & \frac{\partial e}{\partial H} & \frac{\partial e}{\partial l} & \frac{\partial e}{\partial g} & \frac{\partial e}{\partial h} \\ \frac{\partial \omega}{\partial L} & \frac{\partial \omega}{\partial G} & \frac{\partial \omega}{\partial H} & \frac{\partial \omega}{\partial l} & \frac{\partial \omega}{\partial g} & \frac{\partial \omega}{\partial h} \\ \frac{\partial I}{\partial L} & \frac{\partial I}{\partial G} & \frac{\partial I}{\partial H} & \frac{\partial I}{\partial l} & \frac{\partial I}{\partial g} & \frac{\partial I}{\partial h} \\ \frac{\partial \Omega}{\partial L} & \frac{\partial \Omega}{\partial G} & \frac{\partial \Omega}{\partial H} & \frac{\partial \Omega}{\partial l} & \frac{\partial \Omega}{\partial g} & \frac{\partial \Omega}{\partial h} \\ \frac{\partial u}{\partial L} & \frac{\partial u}{\partial G} & \frac{\partial u}{\partial H} & \frac{\partial u}{\partial l} & \frac{\partial u}{\partial g} & \frac{\partial u}{\partial h} \end{pmatrix} \begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{83}$$

$$= \begin{pmatrix} \frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\ \frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{G\tan I} & -\frac{1}{G\sin I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\eta}{eL}\sin f & -\frac{\sin f}{eL} & 0 & \frac{a}{r} & 0 & 0 \end{pmatrix} \begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} \tag{84}$$

ここからは後半部として前半部の逆変換、即ち

$$\begin{gathered}(dL, dG, dH, dl, dg, dh)\\ \downarrow\\ (da, de, d\omega, dI, d\Omega, dl)\\ \downarrow\\ (da, de, d\omega, dI, d\Omega, du)\\ \downarrow\\ (da, de, d\omega, dI, d\Omega, df)\end{gathered}$$

の方向の変換について計算を行う。

$\boxed{(da, de, d\omega, dI, d\Omega, du) \rightarrow (da, de, d\omega, dI, d\Omega, df)}$

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,f)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial f} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial f} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial f} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial f} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial f} \\ \frac{\partial u}{\partial a} & \frac{\partial u}{\partial e} & \frac{\partial u}{\partial \omega} & \frac{\partial u}{\partial I} & \frac{\partial u}{\partial \Omega} & \frac{\partial u}{\partial f} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{85}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial u}{\partial e} & 0 & 0 & 0 & \frac{\partial u}{\partial f} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{86}$$

式 (86) の変換行列に於いては $a, e, \omega, I, \Omega, f$ はすべて独立であるから

$$\frac{\partial f}{\partial e} = 0, \quad \frac{\partial e}{\partial f} = 0 \tag{87}$$

これを用いて式 (4) の第二式を e で偏微分すると

$$\frac{\partial}{\partial e}\cos u = -\sin u \frac{\partial u}{\partial e} = \frac{\partial}{\partial e}(e+\cos f)\cdot(1+e\cos f)^{-1} + (e+\cos f)\frac{\partial}{\partial e}(1+e\cos f)^{-1} \tag{88}$$

$$= \frac{1}{1+e\cos f}\cdot 1 + (e+\cos f)(-1)(1+e\cos f)^{-2}\cdot\cos f \tag{89}$$

$$= \frac{1}{1+e\cos f} - \frac{e+\cos f}{(1+e\cos f)^2}\cos f \tag{90}$$

$$= \frac{1+e\cos f-(e+\cos f)\cos f}{(1+e\cos f)^2} \tag{91}$$

$$= \frac{1+e\cos f - e\cos f - \cos^2 f}{(1+e\cos f)^2} \tag{92}$$

$$= \frac{1-\cos^2 f}{(1+e\cos f)^2} \tag{93}$$

$$= \frac{\sin^2 f}{(1+e\cos f)^2} \quad (\because (4)) \tag{94}$$

$$\therefore \frac{\partial u}{\partial e} = -\frac{1}{\sin u}\frac{\sin^2 f}{(1+e\cos f)^2} \tag{95}$$

$$= -\frac{1}{\sin u}\left(\frac{\sin u}{\eta}\right)^2 \tag{96}$$

$$= -\frac{\sin u}{\eta^2} \tag{97}$$

同様にして式 (4) の第二式を f で偏微分すると

$$\frac{\partial}{\partial f}\cos u = -\sin u \frac{\partial u}{\partial f} = \frac{\partial}{\partial f}(e+\cos f)\cdot(1+e\cos f)^{-1} + (e+\cos f)\frac{\partial}{\partial f}(1+e\cos f)^{-1} \tag{98}$$

$$= \frac{1}{1+e\cos f}(-\sin f) + (e+\cos f)(-1)(1+e\cos f)^{-2}(-e\sin f) \tag{99}$$

$$= \frac{-\sin f}{1+e\cos f} + \frac{1}{(1+e\cos f)^2}(e+\cos f)e\sin f \tag{100}$$

$$= \frac{-\sin f(1+e\cos f) + e\sin f(e+\cos f)}{(1+e\cos f)^2} \tag{101}$$

$$= \frac{-\sin f - e\sin f\cos f + e^2\sin f + e\sin f\cos f)}{(1+e\cos f)^2} \tag{102}$$

$$= -\frac{1-e^2}{(1+e\cos f)^2}\sin f \tag{103}$$

$$\therefore \frac{\partial u}{\partial f} = \left(-\frac{1}{\sin u}\right)\left(-\frac{1-e^2}{(1+e\cos f)^2}\right)\sin f \tag{104}$$

$$= \frac{1}{\sin u}\cdot\frac{\sin f}{1+e\cos f}\cdot\eta\cdot\eta\cdot\frac{1}{1+e\cos f} \tag{105}$$

$$= \frac{1}{\sin u}\cdot\sin u\cdot\eta\cdot\frac{r}{a\eta^2} \tag{106}$$

$$= \frac{r}{a\eta} \tag{107}$$

以上より、式 (86) は以下のようになる。

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,f)}\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} = \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial u}{\partial e} & 0 & 0 & 0 & \frac{\partial u}{\partial f} \end{pmatrix}\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{108}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & -\frac{\sin u}{\eta^2} & 0 & 0 & 0 & \frac{r}{a\eta} \end{pmatrix}\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{109}$$

$(da, de, d\omega, dI, d\Omega, dl) \to (da, de, d\omega, dI, d\Omega, du)$

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} = \frac{\partial(a, e, \omega, I, \Omega, l)}{\partial(a, e, \omega, I, \Omega, u)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial u} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial u} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial u} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial u} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial u} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial u} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{110}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial l}{\partial e} & 0 & 0 & 0 & \frac{\partial l}{\partial u} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{111}$$

式 (111) の変換行列に於いては $a, e, \omega, I, \Omega, u$ はすべて独立であるから

$$\frac{\partial u}{\partial e} = 0, \quad \frac{\partial e}{\partial u} = 0 \tag{112}$$

これにより、ケプラー方程式 Kepler 方程式 (33) の両辺を e で偏微分すると

$$\frac{\partial l}{\partial e} = \frac{\partial}{\partial e}(u - e \sin u) = -\sin u \tag{113}$$

同様にして、ケプラー方程式 Kepler 方程式 (33) の両辺を u で偏微分すると

$$\frac{\partial l}{\partial u} = \frac{\partial}{\partial u}(u - e \sin u) = 1 - e \cos u = \frac{r}{a} \tag{114}$$

以上より、式 (111) は以下のようになる。

$$\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} = \frac{\partial(a, e, \omega, I, \Omega, l)}{\partial(a, e, \omega, I, \Omega, u)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & \frac{\partial l}{\partial e} & 0 & 0 & 0 & \frac{\partial l}{\partial u} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{115}$$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & -\sin u & 0 & 0 & 0 & \frac{r}{a} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{116}$$

$(da, de, d\omega, dI, d\Omega, dl) \to (da, de, d\omega, dI, d\Omega, df)$

この変換は $dl \to du \to df$ と経由して行けば良く、

$$
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix}
= \frac{\partial(a,e,\omega,I,\Omega,l)}{\partial(a,e,\omega,I,\Omega,f)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix}
= \begin{pmatrix}
\frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial f} \\
\frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial f} \\
\frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial f} \\
\frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial f} \\
\frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial f} \\
\frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial f}
\end{pmatrix}
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{117}
$$

$$
= \frac{\partial(a,e,\omega,I,\Omega,l)}{\partial(a,e,\omega,I,\Omega,u)} \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,f)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{118}
$$

$$
= \begin{pmatrix}
1 & & & & \text{\Large 0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\text{\Large 0} & & & & 1 & 0 \\
0 & \frac{\partial l}{\partial e} & 0 & 0 & 0 & \frac{\partial l}{\partial u}
\end{pmatrix}
\begin{pmatrix}
1 & & & & \text{\Large 0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\text{\Large 0} & & & & 1 & 0 \\
0 & \frac{\partial u}{\partial e} & 0 & 0 & 0 & \frac{\partial u}{\partial f}
\end{pmatrix}
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{119}
$$

$$
= \begin{pmatrix}
1 & & & & \text{\Large 0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\text{\Large 0} & & & & 1 & 0 \\
0 & -\sin u & 0 & 0 & 0 & \frac{r}{a}
\end{pmatrix}
\begin{pmatrix}
1 & & & & \text{\Large 0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\text{\Large 0} & & & & 1 & 0 \\
0 & -\frac{\sin u}{\eta^2} & 0 & 0 & 0 & \frac{r}{a\eta}
\end{pmatrix}
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{120}
$$

$$
= \begin{pmatrix}
1 & & & & \text{\Large 0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\text{\Large 0} & & & & 1 & 0 \\
0 & -\sin u - \frac{r}{a}\frac{\sin u}{\eta^2} & 0 & 0 & 0 & -\frac{r}{a}\frac{\sin u}{\eta^2}
\end{pmatrix}
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{121}
$$

$$
= \begin{pmatrix}
1 & & & & \text{\Large 0} & 0 \\
 & 1 & & & & 0 \\
 & & 1 & & & 0 \\
 & & & 1 & & 0 \\
\text{\Large 0} & & & & 1 & 0 \\
0 & -\sin u\left(1 - \frac{r}{a\eta^2}\right) & 0 & 0 & 0 & \frac{r^2}{a^2\eta}
\end{pmatrix}
\begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{122}
$$

$(dL, dG, dH, dl, dg, dh) \to (da, de, d\omega, dI, d\Omega, dl)$

$$\begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} = \frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,l)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{123}$$

$$= \begin{pmatrix} \frac{\partial L}{\partial a} & \frac{\partial L}{\partial e} & \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial I} & \frac{\partial L}{\partial \Omega} & \frac{\partial L}{\partial l} \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial e} & \frac{\partial G}{\partial \omega} & \frac{\partial G}{\partial I} & \frac{\partial G}{\partial \Omega} & \frac{\partial G}{\partial l} \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial e} & \frac{\partial H}{\partial \omega} & \frac{\partial H}{\partial I} & \frac{\partial H}{\partial \Omega} & \frac{\partial H}{\partial l} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial l} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial e} & \frac{\partial g}{\partial \omega} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial \Omega} & \frac{\partial g}{\partial l} \\ \frac{\partial h}{\partial a} & \frac{\partial h}{\partial e} & \frac{\partial h}{\partial \omega} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial \Omega} & \frac{\partial h}{\partial l} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{124}$$

L の定義より

$$\frac{\partial L}{\partial a} = \frac{\partial}{\partial a}\sqrt{\mu a} = \frac{1}{2}\sqrt{\frac{\mu}{a}} = \frac{1}{2}\frac{\mu}{\sqrt{\mu a}} = \frac{\mu}{2L}. \tag{125}$$

L は a にのみ依存するので、

$$\frac{\partial L}{\partial e} = \frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial I} = \frac{\partial L}{\partial \Omega} = \frac{\partial L}{\partial l} = 0. \tag{126}$$

G の定義より

$$\frac{\partial G}{\partial a} = \frac{\partial}{\partial a}\sqrt{\mu a(1-e^2)} = \frac{1}{2}\sqrt{\frac{\mu(1-e^2)}{a}} = \frac{1}{2}\frac{\mu\sqrt{1-e^2}}{\sqrt{\mu a}} = \frac{\mu\eta}{2L}, \tag{127}$$

$$\frac{\partial G}{\partial e} = L\frac{\partial}{\partial e}\sqrt{1-e^2} = L \cdot \frac{1}{2}\left(1-e^2\right)^{-\frac{1}{2}} \cdot (-2e) = -\frac{eL}{\sqrt{1-e^2}} = -\frac{eL}{\eta} = -\frac{eL^2}{G}. \tag{128}$$

G は a, e にのみ依存するので、

$$\frac{\partial G}{\partial \omega} = \frac{\partial G}{\partial I} = \frac{\partial G}{\partial \Omega} = \frac{\partial G}{\partial l} = 0. \tag{129}$$

L, G, H の定義より

$$\frac{\partial H}{\partial a} = \frac{\partial}{\partial a}\left(L\eta\cos I\right) = \frac{\mu}{2L} \cdot \eta\cos I = \frac{\mu\eta}{2L}\cos I, \tag{130}$$

$$\frac{\partial H}{\partial e} = L\frac{\partial}{\partial e}\left(G\cos I\right) = \cos I\frac{\partial G}{\partial e} = -\frac{eL^2}{G}\cos I. \tag{131}$$

$$\frac{\partial H}{\partial I} = G\frac{\partial}{\partial I}\cos I = -G\sin I. \tag{132}$$

H は a, e, I にのみ依存するので、

$$\frac{\partial H}{\partial \omega} = \frac{\partial H}{\partial \Omega} = \frac{\partial H}{\partial l} = 0. \tag{133}$$

式 (124) 左辺の l は Delauney 要素、右辺の l は軌道 6 要素だが、実は同一物である。即ち

$$\frac{\partial l}{\partial l} = 1, \tag{134}$$

$$\frac{\partial l}{\partial a} = \frac{\partial l}{\partial e} = \frac{\partial l}{\partial \omega} = \frac{\partial l}{\partial I} = \frac{\partial l}{\partial \Omega} = 0. \tag{135}$$

変換式 (124) 左辺の g は Delauney 要素だが、実は右辺の $g = \omega$ すなわち近点引数と同一物である。即ち

$$\frac{\partial g}{\partial \omega} = 1, \tag{136}$$

$$\frac{\partial g}{\partial a} = \frac{\partial g}{\partial e} = \frac{\partial g}{\partial I} = \frac{\partial l}{\partial \Omega} = \frac{\partial g}{\partial l} = 0. \tag{137}$$

変換式 (124) 左辺の h は Delauney 要素だが、実は右辺の $h = \Omega$ すなわち昇交点黄経と同一物である。即ち

$$\frac{\partial h}{\partial \Omega} = 1, \tag{138}$$

$$\frac{\partial h}{\partial a} = \frac{\partial h}{\partial e} = \frac{\partial h}{\partial \omega} = \frac{\partial h}{\partial I} = \frac{\partial h}{\partial l} = 0. \tag{139}$$

従って変換式 (124) は

$$\begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} = \frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,l)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{140}$$

$$= \begin{pmatrix} \frac{\partial L}{\partial a} & \frac{\partial L}{\partial e} & \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial I} & \frac{\partial L}{\partial \Omega} & \frac{\partial L}{\partial l} \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial e} & \frac{\partial G}{\partial \omega} & \frac{\partial G}{\partial I} & \frac{\partial G}{\partial \Omega} & \frac{\partial G}{\partial l} \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial e} & \frac{\partial H}{\partial \omega} & \frac{\partial H}{\partial I} & \frac{\partial H}{\partial \Omega} & \frac{\partial H}{\partial l} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial l} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial e} & \frac{\partial g}{\partial \omega} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial \Omega} & \frac{\partial g}{\partial l} \\ \frac{\partial h}{\partial a} & \frac{\partial h}{\partial e} & \frac{\partial h}{\partial \omega} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial \Omega} & \frac{\partial h}{\partial l} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{141}$$

$$= \begin{pmatrix} \frac{\partial L}{\partial a} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial e} & 0 & 0 & 0 & 0 \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial e} & 0 & \frac{\partial H}{\partial I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial l}{\partial l} \\ 0 & 0 & \frac{\partial g}{\partial \omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial h}{\partial \Omega} & 0 \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{142}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ dl \end{pmatrix} \tag{143}$$

$\boxed{(dL, dG, dH, dl, dg, dh) \to (da, de, d\omega, dI, d\Omega, du)}$

この変換は $dL \to dl \to du$ と経由して行けば良く、

$$\frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,u)} = \frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,l)} \frac{\partial(a,e,\omega,I,\Omega,l)}{\partial(a,e,\omega,I,\Omega,u)} \tag{144}$$

$$= \begin{pmatrix} \frac{\partial L}{\partial a} & \frac{\partial L}{\partial e} & \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial I} & \frac{\partial L}{\partial \Omega} & \frac{\partial L}{\partial l} \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial e} & \frac{\partial G}{\partial \omega} & \frac{\partial G}{\partial I} & \frac{\partial G}{\partial \Omega} & \frac{\partial G}{\partial l} \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial e} & \frac{\partial H}{\partial \omega} & \frac{\partial H}{\partial I} & \frac{\partial H}{\partial \Omega} & \frac{\partial H}{\partial l} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial l} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial e} & \frac{\partial g}{\partial \omega} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial \Omega} & \frac{\partial g}{\partial l} \\ \frac{\partial h}{\partial a} & \frac{\partial h}{\partial e} & \frac{\partial h}{\partial \omega} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial \Omega} & \frac{\partial h}{\partial l} \end{pmatrix} \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial u} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial u} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial u} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial u} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial u} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial u} \end{pmatrix} \tag{145}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & & \text{\huge 0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \text{\huge 0} & & & & 1 & 0 \\ 0 & -\sin u & 0 & 0 & 0 & \frac{r}{a} \end{pmatrix} \tag{146}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & -\sin u & 0 & 0 & 0 & \frac{r}{a} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{147}$$

即ち

$$\begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} = \frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,u)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial a} & \frac{\partial L}{\partial e} & \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial I} & \frac{\partial L}{\partial \Omega} & \frac{\partial L}{\partial u} \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial e} & \frac{\partial G}{\partial \omega} & \frac{\partial G}{\partial I} & \frac{\partial G}{\partial \Omega} & \frac{\partial G}{\partial u} \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial e} & \frac{\partial H}{\partial \omega} & \frac{\partial H}{\partial I} & \frac{\partial H}{\partial \Omega} & \frac{\partial H}{\partial u} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial u} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial e} & \frac{\partial g}{\partial \omega} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial \Omega} & \frac{\partial g}{\partial u} \\ \frac{\partial h}{\partial a} & \frac{\partial h}{\partial e} & \frac{\partial h}{\partial \omega} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial \Omega} & \frac{\partial h}{\partial u} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{148}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & -\sin u & 0 & 0 & 0 & \frac{r}{a} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ du \end{pmatrix} \tag{149}$$

$\boxed{(dL, dG, dH, dl, dg, dh) \to (da, de, d\omega, dI, d\Omega, df)}$

この変換は $dL \to dl \to du \to df$ と経由して行けば良く、

$$\frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,f)} = \frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,l)} \frac{\partial(a,e,\omega,I,\Omega,l)}{\partial(a,e,\omega,I,\Omega,u)} \frac{\partial(a,e,\omega,I,\Omega,u)}{\partial(a,e,\omega,I,\Omega,f)} \tag{150}$$

$$= \begin{pmatrix} \frac{\partial L}{\partial a} & \frac{\partial L}{\partial e} & \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial I} & \frac{\partial L}{\partial \Omega} & \frac{\partial L}{\partial l} \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial e} & \frac{\partial G}{\partial \omega} & \frac{\partial G}{\partial I} & \frac{\partial G}{\partial \Omega} & \frac{\partial G}{\partial l} \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial e} & \frac{\partial H}{\partial \omega} & \frac{\partial H}{\partial I} & \frac{\partial H}{\partial \Omega} & \frac{\partial H}{\partial l} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial l} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial e} & \frac{\partial g}{\partial \omega} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial \Omega} & \frac{\partial g}{\partial l} \\ \frac{\partial h}{\partial a} & \frac{\partial h}{\partial e} & \frac{\partial h}{\partial \omega} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial \Omega} & \frac{\partial h}{\partial l} \end{pmatrix} \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial u} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial u} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial u} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial u} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial u} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial u} \end{pmatrix} \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial f} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial f} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial f} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial f} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial f} \\ \frac{\partial u}{\partial a} & \frac{\partial u}{\partial e} & \frac{\partial u}{\partial \omega} & \frac{\partial u}{\partial I} & \frac{\partial u}{\partial \Omega} & \frac{\partial u}{\partial f} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & -\sin u & 0 & 0 & 0 & \frac{r}{a} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial I} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial f} \\ \frac{\partial e}{\partial a} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial \omega} & \frac{\partial e}{\partial I} & \frac{\partial e}{\partial \Omega} & \frac{\partial e}{\partial f} \\ \frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & \frac{\partial \omega}{\partial I} & \frac{\partial \omega}{\partial \Omega} & \frac{\partial \omega}{\partial f} \\ \frac{\partial I}{\partial a} & \frac{\partial I}{\partial e} & \frac{\partial I}{\partial \omega} & \frac{\partial I}{\partial I} & \frac{\partial I}{\partial \Omega} & \frac{\partial I}{\partial f} \\ \frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial I} & \frac{\partial \Omega}{\partial \Omega} & \frac{\partial \Omega}{\partial f} \\ \frac{\partial u}{\partial a} & \frac{\partial u}{\partial e} & \frac{\partial u}{\partial \omega} & \frac{\partial u}{\partial I} & \frac{\partial u}{\partial \Omega} & \frac{\partial u}{\partial f} \end{pmatrix} \tag{151}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & -\sin u & 0 & 0 & 0 & \frac{r}{a} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & & \mathbf{0} & 0 \\ & 1 & & & & 0 \\ & & 1 & & & 0 \\ & & & 1 & & 0 \\ \mathbf{0} & & & & 1 & 0 \\ 0 & -\frac{\sin u}{\eta^2} & 0 & 0 & 0 & \frac{r}{a\eta} \end{pmatrix} \tag{152}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & -\left(1+\frac{r}{a\eta^2}\right)\sin u & 0 & 0 & 0 & \frac{r^2}{a^2\eta} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{153}$$

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$$\begin{pmatrix} dL \\ dG \\ dH \\ dl \\ dg \\ dh \end{pmatrix} = \frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,f)} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial a} & \frac{\partial L}{\partial e} & \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial I} & \frac{\partial L}{\partial \Omega} & \frac{\partial L}{\partial f} \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial e} & \frac{\partial G}{\partial \omega} & \frac{\partial G}{\partial I} & \frac{\partial G}{\partial \Omega} & \frac{\partial G}{\partial f} \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial e} & \frac{\partial H}{\partial \omega} & \frac{\partial H}{\partial I} & \frac{\partial H}{\partial \Omega} & \frac{\partial H}{\partial f} \\ \frac{\partial l}{\partial a} & \frac{\partial l}{\partial e} & \frac{\partial l}{\partial \omega} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial \Omega} & \frac{\partial l}{\partial f} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial e} & \frac{\partial g}{\partial \omega} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial \Omega} & \frac{\partial g}{\partial f} \\ \frac{\partial h}{\partial a} & \frac{\partial h}{\partial e} & \frac{\partial h}{\partial \omega} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial \Omega} & \frac{\partial h}{\partial f} \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{154}$$

$$= \begin{pmatrix} \frac{\partial L}{\partial a} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial G}{\partial a} & \frac{\partial G}{\partial e} & 0 & 0 & 0 & 0 \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial e} & 0 & \frac{\partial H}{\partial I} & 0 & 0 \\ 0 & \frac{\partial l}{\partial e} & 0 & 0 & 0 & \frac{\partial l}{\partial f} \\ 0 & 0 & \frac{\partial g}{\partial \omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial h}{\partial \Omega} & 0 \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{155}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & -\left(1+\frac{r}{a\eta^2}\right)\sin u & 0 & 0 & 0 & \frac{r^2}{a^2\eta} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} da \\ de \\ d\omega \\ dI \\ d\Omega \\ df \end{pmatrix} \tag{156}$$

順変換×逆変換＝単位行列？

以上の検算を兼ね、以下の変換行列を計算してみる。これは原理的に単位行列になるべきである。

$$\frac{\partial(L,G,H,l,g,h)}{\partial(L,G,H,l,g,h)} = \frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,f)}\frac{\partial(a,e,\omega,I,\Omega,f)}{\partial(L,G,H,l,g,h)} \tag{157}$$

$$= \begin{pmatrix} \frac{\mu}{2L} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L} & -\frac{eL^2}{G} & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I & -\frac{eL^2}{G}\cos I & 0 & -G\sin I & 0 & 0 \\ 0 & -\left(1+\frac{r}{a\eta^2}\right)\sin u & 0 & 0 & 0 & \frac{r^2}{a^2\eta} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\ \frac{G^2}{eL^3} & -\frac{G}{eL^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{G\tan I} & -\frac{1}{G\sin I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{G^2}{eL^3}\left(\frac{a}{r}+\frac{L^2}{G^2}\right)\sin f & -\frac{G}{eL^2}\left(\frac{a}{r}+\frac{L^2}{G^2}\right)\sin f & 0 & \frac{a^2\eta}{r^2} & 0 & 0 \end{pmatrix} \tag{158}$$

$$= \begin{pmatrix} \frac{\mu}{2L}\frac{2L}{\mu} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\frac{2L}{\mu} - \frac{e}{L^2}\frac{G^2}{eL^3} & \left(-\frac{eL^2}{G}\right)\left(-\frac{G}{eL^2}\right) & 0 & 0 & 0 & 0 \\ \frac{\mu\eta}{2L}\cos I\frac{2L}{\mu} - \frac{eL^2}{G}\cos I\frac{G^2}{eL^3} & \left(-\frac{eL^2}{G}\cos I\right)\left(-\frac{G}{eL^2}\right) - G\sin I\frac{1}{G\tan I} & -G\sin I\left(-\frac{1}{G\sin I}\right) & 0 & 0 & 0 \\ -\left(1+\frac{r}{a\eta^2}\sin u\right)\frac{G^2}{eL^3} + \frac{r^2}{a^2\eta}\frac{G^2}{eL^3}\left(\frac{a}{r}+\frac{L^2}{G^2}\right)\sin f & -\left(1+\frac{r}{a\eta^2}\sin u\right) + \left(-\frac{G}{eL^2}\right)\left(-\frac{G}{eL^2}\right)\left(\frac{a}{r}+\frac{L^2}{G^2}\right)\sin f & 0 & \frac{r^2}{a^2\eta}\frac{a^2\eta}{r^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

自明でない各成分についてそれぞれ計算を行うと以下のようになる。

$$\frac{\mu}{2L}\frac{2L}{\mu} = 1 \tag{159}$$

$$\frac{\mu\eta}{2L}\frac{2L}{\mu} - \frac{e}{L^2}\frac{G^2}{eL^3} = \eta - \frac{G}{L} = \eta - \eta = 0 \tag{160}$$

$$\left(-\frac{eL^2}{G}\right)\left(-\frac{G}{eL^2}\right) = 1 \tag{161}$$

$$\frac{\mu\eta}{2L}\cos I\frac{2L}{\mu} - \frac{eL^2}{G}\cos I\frac{G^2}{eL^3} = \eta\cos I - \frac{G}{L}\cos I = \eta\cos I - \eta\cos I = 0 \tag{162}$$

$$\left(-\frac{eL^2}{G}\cos I\right)\left(-\frac{G}{eL^2}\right) - G\sin I\frac{1}{G\tan I} = \cos I - \cos I = 0 \tag{163}$$

$$-G\sin I\left(-\frac{1}{G\sin I}\right) = 1 \tag{164}$$

$$\frac{r^2}{a^2\eta}\frac{a^2\eta}{r^2} = 1 \tag{165}$$

また、(3)(4) より

$$\sin u = \frac{\eta\sin f}{1+e\cos f} = \frac{r}{a\eta^2}\cdot\eta\sin f = \frac{r}{a\eta}\sin f \tag{166}$$

なので、

$$\begin{aligned}
&-\left(1+\frac{r}{a\eta^2}\sin u\right)\frac{G^2}{eL^3} + \frac{r^2}{a^2\eta}\frac{G^2}{eL^3}\left(\frac{a}{r}+\frac{L^2}{G^2}\right)\sin f && (167)\\
&= -\left(1+\frac{r}{a\eta^2}\right)\frac{r}{a\eta}\sin f\frac{G^2}{eL^3} + \frac{r}{a^2\eta}\frac{G^2}{eL^3}\left(\frac{a}{r}+\frac{1}{\eta^2}\right)\sin f && (168)\\
&= \frac{G^2}{eL^3}\sin f\left[-\left(1+\frac{r}{a\eta^2}\right)\frac{r}{a\eta} + \frac{r^2}{a^2\eta}\left(\frac{a}{r}+\frac{1}{\eta^2}\right)\right] && (169)\\
&= \frac{G^2}{eL^3}\sin f\left(-\frac{r}{a\eta} - \frac{r^2}{a^2\eta^3} + \frac{r}{a\eta} + \frac{r^2}{a^2\eta^3}\right) && (170)\\
&= 0 && (171)
\end{aligned}$$

$$\begin{aligned}
&-\left(1+\frac{r}{a\eta^2}\sin u\right) + \left(-\frac{G}{eL^2}\right)\left(-\frac{G}{eL^2}\right)\left(\frac{a}{r}+\frac{L^2}{G^2}\right)\sin f && (172)\\
&= -\left(1+\frac{r}{a\eta^2}\right)\frac{r}{a\eta}\sin f\left(-\frac{G}{eL^2}\right) + \frac{r}{a^2\eta}\left(-\frac{G}{eL^2}\right)\left(\frac{a}{r}+\frac{1}{\eta^2}\right)\sin f && (173)\\
&= -\frac{G}{eL^2}\sin f\left[-\left(1+\frac{r}{a\eta^2}\right)\frac{r}{a\eta} + \frac{r^2}{a^2\eta}\left(\frac{a}{r}+\frac{1}{\eta^2}\right)\right] && (174)\\
&= -\frac{G}{eL^2}\sin f\left[-\frac{r}{a\eta} - \frac{r^2}{a^2\eta^3} + \frac{r}{a\eta} + \frac{r^2}{a^2\eta^3}\right] && (175)\\
&= 0 && (176)
\end{aligned}$$

以上より、変換行列 (157) は

$$\frac{\partial(L,G,H,l,g,h)}{\partial(L,G,H,l,g,h)} = \frac{\partial(L,G,H,l,g,h)}{\partial(a,e,\omega,I,\Omega,f)}\frac{\partial(a,e,\omega,I,\Omega,f)}{\partial(L,G,H,l,g,h)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \tag{177}$$

となり、各々の変換行列の計算が正しかったことが示された。 ∎